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CLASSIFICATION OF SCALING LIMITS OF UNIFORM QUADRANGULATIONS WITH A BOUNDARY

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We study noncompact scaling limits of uniform random planar quadrangulations with a boundary when their size tends to infinity. Depending on the asymptotic behavior of the boundary size and the choice of the scaling factor, we observe different limiting metric spaces. Among well-known objects like the Brownian plane or the self-similar continuum random tree, we construct two new one-parameter families of metric spaces that appear as scaling limits: the Brownian half-plane with skewness parameter θ and the infinite-volume Brownian disk of perimeter σ. We also obtain various coupling and limit results clarifying the relation between these objects.

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1. Introduction. In this work, we obtain a complete classification of possible scaling limits of finite random planar quadrangulations with a boundary when their size tends to infinity.

Recall that a *planar map* is a proper embedding of a finite connected graph in the two-dimensional sphere. The graph may have loops and multiple edges. The faces of a map are the connected components of the complement of its edges. A planar *quadrangulation with a boundary* is a particular planar map where its faces have degree four, that is, are incident to four edges (an edge is counted twice if it lies entirely in the face), except possibly one distinguished face which may have an arbitrary (even) degree. This face is referred to as the external face, whereas the other faces are called internal faces. The boundary of the map is given by the edges that are incident to the external face, and the number of such edges is called the size of the boundary, or the *perimeter* of the map. The size of the map is given by the number of its internal faces. We do not demand that the boundary forms a simple curve. We always consider rooted maps with a boundary, which means that we distinguish one oriented edge of the boundary such that the root face lies to the left of that edge. This edge will be called the root edge, and its origin the root vertex. As usual, two (rooted) maps are considered equivalent if they differ by an orientation- and root-preserving homeomorphism of the sphere.

We are interested in scaling limits of planar maps picked uniformly at random among all quadrangulations with a boundary when the size and (possibly) the perimeter of the map tend to infinity. This means that we view the vertex set of the quadrangulation as a metric space for the graph distance and consider (under a suitable rescaling of the distance) distributional limits of such spaces, either in the global or local Gromov–Hausdorff topology.

In [30] and independently in [34], it was shown that uniformly chosen quadrangulations of size n, equipped with the graph distance d_{gr} rescaled by a factor $n^{-1/4}$, converge to a random compact metric space called the Brownian map. The latter turns out to be a universal object which appears as the distributional limit of many classes of random maps. We refer to the recent overview [35] for various aspect of the Brownian map and for more references.

Here, we shall deal with quadrangulations of size n having a boundary of size $2\sigma_n$, and we will distinguish three boundary regimes as n tends to infinity:

(a) $\sigma_n/\sqrt{n} \to 0$;
(b) $\sigma_n/\sqrt{n} \to \sqrt{2}\sigma$ for some $\sigma \in (0, \infty)$;
(b) $\sigma_n/\sqrt{n} \to \infty$.

Bettinelli [9] showed that in regime (a), the boundary becomes negligible in the scale $n^{-1/4}$, and the Brownian map appears in the limit when n tends to infinity. In regime (b), he obtained under the same rescaling convergence along appropriate infinite subsequences to a random metric space called the Brownian disk BD_σ. Uniqueness of this limit was later established by Bettinelli and Miermont in [11]. For the third regime (c), it is shown in [9] that a rescaling by $\sigma_n^{-1/2}$ leads in the limit to Aldous' continuum random tree CRT [1, 2].

The scaling factors considered by Bettinelli [9] ensure that the diameter of the rescaled planar map stays bounded in probability. Consequently, the limits he obtains are random compact metric spaces, and the right notion of convergence is the Gromov–Hausdorff convergence in the space of (isometry classes of) compact metric spaces.

We will study all possible scalings $a_n \to \infty$ in all the above boundary regimes, meaning that we replace the graph distance d_{gr} by $a_n^{-1} d_{\mathrm{gr}}$ and take the limit $n \to \infty$. When a_n grows slower than the diameter of the map as n tends to infinity, the right notion of convergence is the *local* Gromov–Hausdorff convergence. Depending on the ratio of perimeter and scaling parameter, the boundary will in the limit be either invisible, or of a size comparable to the full map, or dominate the map.

In the process, we obtain two new one-parameter families of limit spaces: the Brownian half-plane BHP_θ with parameter $\theta \in [0, \infty)$ and the infinite-volume Brownian disk IBD_σ with boundary length $\sigma \in (0, \infty)$. The Brownian disk BD_σ and the Brownian half-plane $\mathsf{BHP} = \mathsf{BHP}_0$ play a central role in this work. The latter can be seen as the Gromov–Hausdorff tangent cone in distribution of BD_σ at its root, and also as the scaling limit of the so-called uniform infinite half-planar quadrangulation UIHPQ. The space BHP_θ for $\theta > 0$ can be understood as an interpolation between BHP (when $\theta \to 0$) and the so-called self-similar continuum random tree SCRT introduced by Aldous [1] (when $\theta \to \infty$). The IBD_σ in turn interpolates between BHP (when $\sigma \to \infty$) and the Brownian plane BP introduced by Curien and Le Gall [20, 21] (when $\sigma \to 0$).

We begin with a rough overview of our main results on scaling limits of finite-size quadrangulations with a boundary (including results of [9] and [11]). We then mention further results that will be obtained below, including limit statements on BD_σ. The precise formulations can be found in Section 3, after a proper definition of the limit spaces and a reminder on the notion of convergence in Section 2.

As in many works in this context, our approach is based on the Bouttier–Di Francesco–Guitter bijection [13, 14], which establishes a one-to-one correspondence between (finite-size) quadrangulations with a boundary on the one hand and discrete labeled forests and bridges on the other hand. The bijection is recalled in Section 4. Section 5 contains some more auxiliary results, mostly convergence results on forests and bridges when their size tends to infinity. The statements proved there are of some independent interest, but can also be skipped at first reading. Section 6 contains all the proofs of our main statements.

1.1. *Overview over the main results.* For any $\sigma_n \in \mathbb{N} = \{1, 2, \dots\}$, we write $Q_n^{\sigma_n}$ for a uniformly distributed rooted quadrangulation with n inner faces and a boundary of size $2\sigma_n$. The vertex set of $Q_n^{\sigma_n}$ is denoted $V(Q_n^{\sigma_n})$, ρ_n represents the root vertex and d_{gr} stands for the graph distance on $V(Q_n^{\sigma_n})$. For any two sequences $(a_n, n \in \mathbb{N})$, $(b_n, n \in \mathbb{N})$ of reals, we write $a_n \ll b_n$ or $b_n \gg a_n$ if and only if $a_n/b_n \to 0$ as $n \to \infty$, and we write $a_n \sim b_n$ if $a_n/b_n \to 1$.

We denote by $\circ$ the trivial one-point metric space and write $\mathsf{s\text{-}Lim}$ ($\mathsf{s\text{-}Lim_{loc}}$) for the scaling limit in law of $(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n)$ in the Gromov–Hausdorff topology (in the local Gromov–Hausdorff topology) as n tends to infinity.

Regime $\sigma_n \ll \sqrt{n}$.

- If $1 \ll a_n \ll \sqrt{\sigma_n}$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{BHP}$.
- If $1 \ll a_n \sim (1/9)^{1/4}\sqrt{2\sigma_n/\sigma}$, $\sigma \in (0, \infty)$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{IBD}_\sigma$.
- If $\sqrt{\sigma_n} \ll a_n \ll n^{1/4}$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{BP}$.
- If $a_n \sim (8/9)^{1/4} n^{1/4}$, then (see [9]) $\mathsf{s\text{-}Lim} = \mathsf{BM}$.
- If $a_n \gg n^{1/4}$, then $\mathsf{s\text{-}Lim} = \circ$.

Regime $\sigma_n \sim \sigma\sqrt{2n}$, $\sigma \in (0, \infty)$.

- If $1 \ll a_n \ll n^{1/4}$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{BHP}$.
- If $a_n \sim (8/9)^{1/4} n^{1/4}$, then (see [9] and [11]) $\mathsf{s\text{-}Lim} = \mathsf{BD}_\sigma$.
- If $a_n \gg n^{1/4}$, then $\mathsf{s\text{-}Lim} = \circ$.

Regime $\sigma_n \gg \sqrt{n}$.

- If $\sigma_n \ll n$ and $\lim_{n\to\infty} (9/4)^{1/4} a_n/\sqrt{2n/\sigma_n} = \sqrt{\theta} \in [0, \infty)$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{BHP}_\theta$.
- If $\max\{1, \sqrt{n/\sigma_n}\} \ll a_n \ll \sqrt{\sigma_n}$, then $\mathsf{s\text{-}Lim_{loc}} = \mathsf{SCRT}$.
- If $a_n \sim \sqrt{2\sigma_n}$ (see [9]), then $\mathsf{s\text{-}Lim} = \mathsf{CRT}$.
- If $a_n \gg \sqrt{\sigma_n}$, then $\mathsf{s\text{-}Lim} = \circ$.

The new results in these listings are covered by Theorems 3.1, 3.2, 3.3, 3.4 and 3.5 below. In the regime $\sigma_n \ll \sqrt{n}$ in the first list, the last three convergences include the case of bounded σ_n. In the last regime $\sigma_n \gg \sqrt{n}$, we allow σ_n to grow faster than n. The scaling constants are chosen in such a way that the description of the limiting objects is the most natural.

Figure 1 shows all possible regimes in one diagram, in which the x-axis denotes the limiting possible values for the logarithm of the boundary length $\log(\sigma_n)/\log(n)$ in units of $\log(n)$, and the y-axis corresponds to the limit of the logarithm of the scaling factor $\log(a_n)/\log(n)$ in units of $\log(n)$. For the specific value $y = 0$, it will be assumed that $a_n = 1$, so that we are in the regime of local limits with no rescaling. Similarly, for some specific values of (x, y), that are shown on the colored lines, we will require some particular scaling behaviors that are detailed in the list above. For instance, for $x = 1/2$ and $y = 1/4$, we really ask that $\sigma_n \sim \sigma\sqrt{2n}$ for some $\sigma > 0$ and $a_n \sim (8/9)^{1/4} n^{1/4}$. Note that the portion $x \geq 1$ of the $y = 0$ axis has been left hashed: indeed it corresponds to a regime of unrescaled local limits, which are studied in [4].

As it is shown in Theorem 3.6, the BHP can also be obtained from the UIHPQ by zooming-out around the root: $\lambda \cdot \mathsf{UIHPQ} \to \mathsf{BHP}$ in distribution in the local Gromov–Hausdorff sense as $\lambda \to 0$. Here, $\lambda \cdot \mathsf{UIHPQ}$ is obtained from UIHPQ by

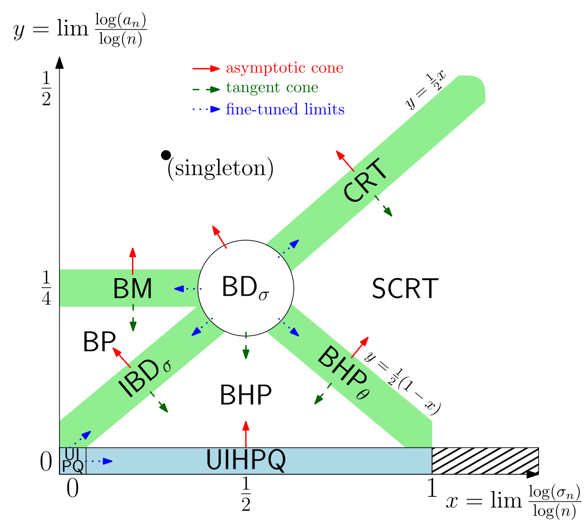

FIG. 1. *The user's manual to this paper, displaying all possible regimes and limits for the rescaled pointed space* $(V(Q_n^{\sigma_n}), a_n^{-1}d_{\mathrm{gr}}, \rho_n)$. *Taking the asymptotic cone (tangent cone) of a pointed space refers to zooming-out (zooming-in) around the distinguished point. We refer to the statements of the results in Section* 3 *for the precise meaning.*

keeping the same set of points, but rescaling the metric by a factor λ; see Section 2.4.2 below.

Many of our results, for example, those involving the Brownian half-planes BHP_θ, $\theta \geq 0$, are based on coupling methods, which yield in fact stronger statements than those mentioned above. In particular, couplings will allow us to determine the topologies of BHP_θ and IBD_σ (Corollaries 3.8 and 3.13).

The above results will moreover enable us to determine the limiting behavior of the Brownian disk $\mathsf{BD}_{T,\sigma}$ with volume T and perimeter σ when zooming-in around its root vertex, or, equivalently by scaling, by blowing up its volume and perimeter. Depending on the behavior of the "perimeter" function $\sigma(\cdot) : (0,\infty) \to (0,\infty)$ for large volumes T, we observe BP, IBD_ς, BHP_θ or the SCRT as the distributional limit in the local Gromov–Hausdorff sense of $\mathsf{BD}_{T,\sigma(T)}$ when $T \to \infty$; see Figure 4 below and Corollary 3.15.

2. Definitions. In this section, we define our limit objects and recall some facts about the (local) Gromov–Hausdorff convergence.

All our limit metric spaces will be defined in terms of certain random processes. To make the presentation unified, we will denote by (X, W) the canonical continuous process in $\mathcal{C}(I, \mathbb{R})^2$, where I will always denote an interval of the form $I = [0, T]$ for some $T > 0$, or $I = \mathbb{R}$. In the definitions to come, when we say, for instance, that X is a Brownian motion, we will really mean that X is considered under the law of Brownian motion. The set $\mathcal{C}(I, \mathbb{R})$ of continuous functions on I is equipped with the compact-open topology (topology of uniform convergence over compact subsets of I). For reasons that will become clear later on, we will often refer to X as the contour process, whereas W will be called the label process.

For $t \in I \cap [0, \infty)$, we write $\underline{X}_t = \inf_{[0,t]} X$, and in case $I = \mathbb{R}$, we put for $t < 0$ $\underline{X}_t = \inf_{(-\infty,t]} X$.

If $Y = (Y_t, t \geq 0)$ is a real-valued process indexed by the positive real half-line, we write $\Pi(Y)$ for its *Pitman transform* defined as $\Pi(Y)_t = Y_t - 2\underline{Y}_t$, $t \geq 0$. We will often use the fact that if $B = (B_t, t \geq 0)$ is a standard Brownian motion, then its Pitman transform $\Pi(B)$ has the law of a three-dimensional Bessel process, and $\inf_{[t,\infty)} \Pi(B) = -\inf_{[0,t]} B$ for every $t \geq 0$; see [36], Theorem 0.1(ii).

2.1. *Metric spaces coded by real functions.*

Real trees. Let $f \in \mathcal{C}(I, \mathbb{R})$. For $s, t \in I$, we denote by $\underline{f}(s,t)$ the quantity

$$\underline{f}(s,t) = \begin{cases} \inf\limits_{[s,t]} f & \text{if } s \leq t, \\ \inf\limits_{I \setminus (t,s)} f & \text{if } s > t, \end{cases}$$

and for $s, t \in I$ we let

$$d_f(s,t) = f(s) + f(t) - 2\max\{\underline{f}(s,t), \underline{f}(t,s)\}. \tag{2.1}$$

The function d_f defines a pseudo-metric on I, which is a class function for the equivalence relation $\{d_f = 0\}$. Therefore, we can define the quotient space $\mathcal{T}_f = I/\{d_f = 0\}$, on which d_f induces a true distance, still denoted by d_f for simplicity. Since we assumed that I contains 0, it is natural to "root" the space $(\mathcal{T}_f, d_f)$ at the point ρ given by the equivalence class $[0] = \{s \in I : d_f(0,s) = 0\}$ of 0.

The metric space $(\mathcal{T}_f, d_f, \rho)$ is called the *continuum tree coded by* f. In more precise terms, it is a rooted $\mathbb{R}$-tree, which is also compact if I is compact. This fact is well known in the "classical case" where f is a nonnegative function on an interval $[0, T]$, and $f(0) = f(T) = 0$; see, for example, [31], Section 3, and it remains true in our more general context.

Note that the space $(\mathcal{T}_f, d_f)$ comes with a natural Borel σ-finite measure, μ_f, which is defined as the push-forward of the Lebesgue measure on I by the canonical projection $p_f : I \to \mathcal{T}_f$.

Metric gluing of a real tree on another. Let f, g be two elements of $\mathcal{C}(I, \mathbb{R})$. These functions code two $\mathbb{R}$-trees $\mathcal{T}_f, \mathcal{T}_g$ in the preceding sense. We define a new metric space $(M_{f,g}, D_{f,g})$ by informally quotienting the space $(\mathcal{T}_g, d_g)$ by the equivalence relation $\{d_f = 0\}$. Formally, for $s, t \in I$, we let $D_{f,g}(s,t)$ be given by

$$\inf\left\{\sum_{i=1}^{k} d_g(s_i, t_i) : \begin{array}{l} k \geq 1, s_1, \ldots, s_k, t_1, \ldots, t_k \in I, s_1 = s, t_k = t, \\ d_f(t_i, s_{i+1}) = 0 \text{ for every } i \in \{1, 2, \ldots, k-1\} \end{array}\right\}. \tag{2.2}$$

This defines a pseudo-metric on I, and we let $M_{f,g}$ be the quotient space $I/\{D_{f,g}=0\}$, endowed with the true metric inherited from $D_{f,g}$ (and again, still denoted by $D_{f,g}$). Again, the space $(M_{f,g}, D_{f,g})$ is naturally pointed at the equivalence class of 0 for $\{D_{f,g}=0\}$ (which we still denote by ρ), and naturally endowed with the measure $\mu_{f,g}$, defined as the push-forward of the Lebesgue measure on I by the canonical projection $p_{f,g}: I \to M_{f,g}$.

Note that in the classical definition of the Brownian map and related objects, one has the extra property that $d_f(s,t)=0$ implies that $g(s)=g(t)$, and this will indeed always be the case in all concrete cases considered in this paper. However, the definition makes sense without this assumption.

2.2. *Random snakes*. The definition of most of our limiting random spaces depend on the notion of a random snake, which we introduce next. Let $f \in \mathcal{C}(I,\mathbb{R})$ be a continuous path on an interval I satisfying $f(0)=f(T)$ in case $I=[0,T]$. The random snake driven by f is a centered Gaussian process $(Z_s^f, s\in I)$ satisfying $Z_0^f=0$ a.s. and

$$\mathbb{E}\big[\big|Z_s^f - Z_t^f\big|^2\big] = d_f(s,t).$$

These specifications characterize the law of Z^f: roughly speaking, it can be seen as Brownian motion indexed by the tree $\mathcal{T}_f$; see, for example, Section 4 of [31]. It is easy to see and well known that the process Z^f admits a continuous modification as soon as f is a locally Hölder-continuous function on I. In this case, we always work with this modification.

We will consider random snakes driven by random functions. The snake driven by a random function Y is simply defined as the random Gaussian process Z^Y conditionally given Y. In all our applications, Y will be considered under probability distributions that make it a Hölder-continuous function with probability one. Moreover, Y will almost surely satisfy $Y_0=Y_T=0$ in the two cases where $I=[0,T]$ (namely for the Brownian map and disk).

2.3. *Limit random metric spaces*. We apply the preceding constructions to a variety of random versions of the functions f, g.

2.3.1. *Compact spaces*. In this section, the processes considered all take values in $\mathcal{C}([0,T],\mathbb{R})$ for some $T>0$.

DEFINITION 2.1. Let $T>0$. The *continuum random tree* CRT_T with volume T is the real tree $(\mathcal{T}_X, d_X, \rho)$ where $X=(X_t, t\in[0,T])$ is a Brownian excursion with duration T.

The CRT_T was introduced by Aldous [1, 2]. We simply write CRT instead of CRT_1. Note the scaling relation $\lambda\cdot\mathsf{CRT}_T =_d \mathsf{CRT}_{\lambda^2 T}$ for $\lambda>0$. This comes from the fact that if $\mathbb{e}^T$ is a Brownian excursion with duration T, then $\lambda\mathbb{e}^T(\cdot/\lambda^2)$ has same distribution as $\mathbb{e}^{\lambda^2 T}$. We stress that the point ρ plays no distinguished role in the above construction. Indeed, roughly speaking, the rerooting property of CRT_T [2], (20), states that if ρ' is distributed according to $\mu_X/\mu_X(1)$ (the normalized version of the measure defined above), then $(\mathcal{T}_X, d_X, \rho')$ has same law as $(\mathcal{T}_X, d_X, \rho)$.

DEFINITION 2.2. Let $T > 0$. The *Brownian map* BM_T with volume T is the metric space $(M_{X,W}, D_{X,W}, \rho)$ where X is a Brownian excursion of duration T, and W is the snake driven by X.

See [30, 34] for a description of the Brownian map. We write BM instead of BM_1. The scaling properties of Gaussian processes imply that for $\lambda > 0$, $\lambda \cdot \mathsf{BM}_T =_d \mathsf{BM}_{\lambda^4 T}$. Just as for CRT_T, the point ρ in BM_T should be seen as a random choice according to the normalized measure $\mu_{X,W}/\mu_{X,W}(1)$, which is known as the rerooting property of the Brownian map (Theorem 8.1 of [29]).

DEFINITION 2.3. Let $T > 0$, $\sigma > 0$. The *Brownian disk* $\mathsf{BD}_{T,\sigma}$ with volume T and boundary length σ is the metric space $(M_{X,W}, D_{X,W}, \rho)$ where X is a first passage Brownian bridge from 0 to $-\sigma$ of duration T, and conditionally given X, $(W_t, 0 \le t \le T)$ has same distribution as $(\sqrt{3}\gamma_{-\underline{X}_t} + Z_t, 0 \le t \le T)$, where:

- $(Z_t, 0 \le t \le T) = Z^{X-\underline{X}}$ is the snake driven by the process $(X_t - \underline{X}_t, 0 \le t \le T)$;
- $(\gamma_x, 0 \le x \le \sigma)$ is a Brownian bridge with duration σ, independent of $Z^{X-\underline{X}}$.

The Brownian disk has first been constructed in [9, 11]. Note that the conditional covariances of the snake $Z^{X-\underline{X}}$ are given by

$$\mathbb{E}[Z_s Z_t \mid X] = \min_{[s,t]}(X - \underline{X}), \qquad 0 \le s \le t \le T.$$

If $T = 1$, we will simply write BD_σ instead of $\mathsf{BD}_{1,\sigma}$. The Brownian disks are homeomorphic to the closed unit disk $\overline{\mathbb{D}}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$; see [9], Proposition 21 (cited as Lemma 6.11 below). They enjoy the following scaling property: For $\lambda > 0$, $\lambda \cdot \mathsf{BD}_{T,\sigma} =_d \mathsf{BD}_{\lambda^4 T, \lambda^2\sigma}$. Contrary to the Brownian tree or the Brownian map, ρ does not play the role of a random point distributed according to $\mu_{X,W}/\mu_{X,W}(1)$. The reason is that ρ is a.s. a point of the boundary of the disk, which is of zero measure (see [11] for more details).

2.3.2. *Noncompact spaces.* In this section, all processes take values in $\mathcal{C}(\mathbb{R}, \mathbb{R})$.

DEFINITION 2.4. The *self-similar continuum random tree* SCRT is the real tree $(\mathcal{T}_X, d_X, \rho)$ where $X = (X_t, t \in \mathbb{R})$ is such that $(X_t, t \ge 0)$ and $(X_{-t}, t \ge 0)$ are two independent three-dimensional Bessel processes started at 0.

The SCRT appears as process 2 in [1]. It fulfills the self-similarity property $\lambda \cdot \mathsf{SCRT} =_d \mathsf{SCRT}$ for every $\lambda > 0$. Note that if we let Y be the canonical process in $\mathcal{C}(\mathbb{R}, \mathbb{R})$ under the probability law which turns $(Y_t, t \geq 0)$ and $(Y_{-t}, t \geq 0)$ into two independent Brownian motions, then $(\mathcal{T}_Y, d_Y, [0])$ has same law as SCRT. This follows readily from the fact that $\Pi((Y_t, t \geq 0))$ has the law of a three-dimensional Bessel process.

DEFINITION 2.5. The *Brownian plane* BP is the metric space given by $(M_{X,W}, D_{X,W}, \rho)$ where:

- $(X_t, t \geq 0)$ and $(X_{-t}, t \geq 0)$ are two independent three-dimensional Bessel processes;
- given $X = (X_t, t \in \mathbb{R})$, W has same distribution as the snake Z^X driven by X.

The Brownian plane was introduced in [20] (see also [15] for a hyperbolic version). It is a.s. homeomorphic to $\mathbb{R}^2$ and invariant under scaling, in the sense that for $\lambda > 0$, $\lambda \cdot \mathsf{BP} =_d \mathsf{BP}$.

DEFINITION 2.6. Let $\theta \geq 0$. The *Brownian half-plane* BHP_θ with skewness parameter θ is the metric space $(M_{X,W}, D_{X,W}, \rho)$ where:

- $(X_t, t \geq 0)$ is a Brownian motion with linear drift $-\theta$, and $(X_{-t}, t \geq 0)$ is the Pitman transform $\Pi(X')$ of an independent copy X' of $(X_t, t \geq 0)$;
- given X, W has same distribution as $(\sqrt{3}\gamma_{-\underline{X}_t} + Z_t, t \in \mathbb{R})$, where:
  - $(Z_t, t \in \mathbb{R}) = Z^{X-\underline{X}}$ is the snake driven by the process $(X_t - \underline{X}_t, t \in \mathbb{R})$;
  - $(\gamma_x, x \in \mathbb{R})$ is a two-sided Brownian motion with $\gamma_0 = 0$, independent of $Z^{X-\underline{X}}$.

The Brownian half-planes are the first truly new limiting metric spaces that we encounter in this study. The space BHP_θ enjoys the scaling property $\lambda \cdot \mathsf{BHP}_\theta =_d \mathsf{BHP}_{\theta/\lambda^2}$ for $\lambda > 0$. This makes the value $\theta = 0$ special in the sense that BHP_0 is self-similar in law (just as SCRT or BP), and we shall often write BHP instead of BHP_0. We will see in Corollary 3.8 that for every $\theta \geq 0$, BHP_θ is a.s. homeomorphic to the closed half-plane $\overline{\mathbb{H}} = \mathbb{R} \times \mathbb{R}_+$.

REMARK 2.7. A random metric space called the Brownian half-plane first appeared in the recent work [17], where it is conjectured to arise as the scaling limit of the uniform infinite half-planar quadrangulation UIHPQ; see Section 4.4. Theorem 3.6 below states indeed that the scaling limit of UIHPQ is the space BHP_0. However, the definition of the Brownian half-plane from [17] differs from ours: it is still of the form $(M_{X,W}, D_{X,W}, \rho)$, but for processes (X, W) having a very different law from that of Definition 2.6 (with $\theta = 0$). We do not actually prove that the two definitions coincide, since we believe that this would require some specific work. Nonetheless, we prefer to stick to the name "Brownian half-plane" since we feel that this should be the proper denomination for the scaling limit of the UIHPQ.

DEFINITION 2.8. Let $\sigma > 0$. The *infinite-volume Brownian disk* IBD_σ with boundary length σ is the metric space $(M_{X,W}, D_{X,W}, \rho)$ where:

- $(X_t, t \in \mathbb{R})$ has the law of $(V_{t-L} - U, t \in \mathbb{R})$, where $(V_t, t \geq 0)$ and $(V_{-t}, t \geq 0)$ are two independent three-dimensional Bessel processes with $V_0 = 0$, U is uniform on $[0, \sigma]$ and independent of V, and $L = \sup\{t \geq 0 : V_{-t} = U\}$;
- given X, W has same distribution as $(\sqrt{3}\gamma_{-\underline{\underline{X}}_t^\sigma} + Z_t, t \in \mathbb{R})$, where:
  –
$$\underline{\underline{X}}_t = \begin{cases} \min\left\{ \inf\limits_{(-\infty,t]} X, \inf\limits_{[0,\infty)} X + \sigma \right\} & \text{if } t \leq 0, \\ \min\limits_{[0,t]} X & \text{if } t \geq 0 \end{cases}$$
  and $\underline{\underline{X}}^\sigma = \underline{\underline{X}} - \sigma$ on $(-\infty, 0)$, $\underline{\underline{X}}^\sigma = \underline{\underline{X}}$ on $[0, \infty)$;
  – $(Z_t, t \in \mathbb{R}) = Z^{X - \underline{\underline{X}}}$ is the random snake driven by the process $X - \underline{\underline{X}}$;
  – $(\gamma_x, 0 \leq x \leq \sigma)$ is a Brownian bridge with duration σ, independent of $Z^{X-\underline{\underline{X}}}$.

The infinite-volume Brownian disk should be thought of as a Brownian disk with perimeter σ filled in with a Brownian plane BP; see Remark 2.9 below. It enjoys the scaling property $\lambda \cdot \mathsf{IBD}_\sigma =_d \mathsf{IBD}_{\lambda^2\sigma}$ for $\lambda > 0$. We will prove in Corollary 3.13 that for every $\sigma > 0$, IBD_σ is a.s. homeomorphic to the pointed closed disk $\overline{\mathbb{D}} \setminus \{0\}$.

REMARK 2.9. We give an equivalent description of the contour process X under the law of the infinite-volume Brownian disk IBD_σ, which will be useful for our purpose. Let $(B_t, t \geq 0)$ be a Brownian motion with $B_0 = 0$, $T_x = \inf\{t \geq 0 : B_t < -x\}$ the first hitting time of $(-\infty, -x)$, R, R' two independent three-dimensional Bessel processes independent of B, and U_0 a uniform random variable in $[0, \sigma]$, independent of B, R, R'. Letting

$$Y_t^\sigma = \begin{cases} R'_{-t+T_{U_0}-T_\sigma} + \sigma - U_0 & \text{if } t \leq T_{U_0} - T_\sigma, \\ B_{T_\sigma + t} + \sigma & \text{if } T_{U_0} - T_\sigma \leq t \leq 0, \\ B_t & \text{if } 0 \leq t \leq T_{U_0}, \\ -U_0 + R_{t - T_{U_0}} & \text{if } t \geq T_{U_0}, \end{cases}$$

William's time-reversal theorem (see, e.g., (0.29) of [36]) entails that $(Y_t^\sigma, t \in \mathbb{R})$ has same law as the canonical process $(X_t, t \in \mathbb{R})$ under the law of IBD_σ. Intuitively, at time T_{U_0}, the encoding of a Brownian plane in terms of the Bessel processes R and R' "inside" a (free pointed) Brownian disk with boundary length

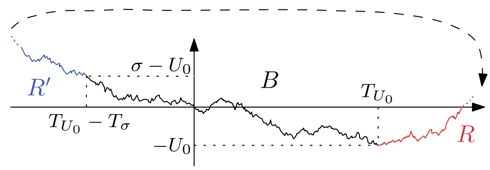

FIG. 2. *The contour process* $(Y_t^\sigma, t \in \mathbb{R})$ *of the infinite-volume Brownian disk* IBD_σ.

σ starts. The contour process of the latter is given by $(Y_t^\sigma, T_{U_0} - T_\sigma \le t \le T_{U_0})$. The denomination "free" means that the volume of the disk is not fixed; we refer to [11], Section 1.5, for a precise definition. An illustration is shown in Figure 2.

Finally, we will encounter the *uniform infinite half-planar quadrangulation* UIHPQ $Q_\infty^\infty = (V(Q_\infty^\infty), d_{\mathrm{gr}}, \rho)$, which is an infinite rooted random quadrangulation with an infinite boundary. It arises as the distributional limit of $Q_n^{\sigma_n}$, $1 \ll \sigma_n \ll n$, for the so-called local metric d_{map}; see Proposition 3.11. We defer to Section 2.4.3 for a definition of the metric and to Section 4.4 for a precise construction of the UIHPQ.

2.4. *Notions of convergence.*

2.4.1. *Gromov–Hausdorff convergence.* Given two pointed compact metric spaces $\mathbf{E} = (E, d, \rho)$ and $\mathbf{E}' = (E', d', \rho')$, the Gromov–Hausdorff distance between $\mathbf{E}$ and $\mathbf{E}'$ is given by

$$d_{\mathrm{GH}}(\mathbf{E}, \mathbf{E}') = \inf\{d_{\mathrm{H}}(\varphi(E), \varphi'(E)) \vee \delta(\varphi(\rho), \varphi'(\rho'))\},$$

where the infimum is taken over all isometric embeddings $\varphi : E \to F$ and $\varphi' : E' \to F$ of E and E' into the same metric space (F, δ), and d_{H} denotes the Hausdorff distance between compact subsets of F. The space of all isometry classes of pointed compact metric spaces $(\mathbb{K}, d_{\mathrm{GH}})$ forms a Polish space.

An alternative characterization of the Gromov–Hausdorff distance can be obtained *via* correspondences. A *correspondence* between two pointed metric spaces $\mathbf{E} = (E, d, \rho)$, $\mathbf{E}' = (E', d', \rho')$ is a subset $\mathcal{R} \subset E \times E'$ such that $(\rho, \rho') \in \mathcal{R}$, and for every $x \in E$ there exists at least one $x' \in E'$ such that $(x, x') \in \mathcal{R}$ as well as for every $y' \in E'$, there exists at least one $y \in E$ such that $(y, y') \in \mathcal{R}$. The distortion of $\mathcal{R}$ with respect to d and d' is given by

$$\mathrm{dis}(\mathcal{R}) = \sup\{|d(x, y) - d'(x', y')| : (x, x'), (y, y') \in \mathcal{R}\}.$$

Then it holds that (see, e.g., [16], Theorem 7.3.25)

$$d_{\mathrm{GH}}(\mathbf{E}, \mathbf{E}') = \frac{1}{2} \inf_{\mathcal{R}} \mathrm{dis}(\mathcal{R}),$$

where the infimum is taken over all correspondences between $\mathbf{E}$ and $\mathbf{E}'$.

The convergences listed in the overview above which involve compact limiting spaces, that is, BM, BD_σ, CRT and the trivial one-point space, hold in distribution in $(\mathbb{K}, d_{\mathrm{GH}})$.

2.4.2. *Local Gromov–Hausdorff convergence*. Noncompact limits like the spaces BHP_θ, IBD_σ or SCRT will be obtained in the local Gromov–Hausdorff topology. Roughly speaking, local Gromov–Hausdorff convergence requires only convergence of balls of a fixed radius seen as compact metric spaces.

We give only a quick reminder; for more details, we refer to Chapter 8 of [16]. As in [20], we can restrict ourselves to the case of (pointed) complete and locally compact length spaces (see our discussion below). Recall that a metric space (E, d) is a *length space* if for every pair (x, y) of points in E, the distance $d(x, y)$ agrees with the infimum over the lengths of all continuous paths from x to y. Here, a continuous path from x to y is a continuous function $\gamma : [0, T] \to E$ with $\gamma(0) = x$ and $\gamma(T) = y$ for some $T \geq 0$, and the length of γ is given by

$$L(\gamma) = \sup_\tau \sum_{k=1}^{n-1} d\big(\gamma(t_k), \gamma(t_{k+1})\big),$$

where the supremum is taken over all subdivisions τ of $[0, T]$ of the form $0 = t_1 < t_2 < \cdots < t_n = T$ for some $n \in \mathbb{N}$. A path γ for which the infimum over the length is attained is called a *geodesic*. Note that in a complete and locally compact length space (E, d), any two points $x, y \in E$ with $d(x, y) < \infty$ are joined by a geodesic; see [16], Theorem 2.5.23.

Now let $\mathbf{E} = (E, d, \rho)$ be a pointed metric space, that is, a metric space with a distinguished point $\rho \in E$. We denote by $B_r(\mathbf{E})$ the closed ball of radius r around ρ in $\mathbf{E}$. Equipped with the restriction of d, we view $B_r(\mathbf{E})$ as a pointed compact metric space.

Given pointed complete and locally compact length spaces $(\mathbf{E}_n)_n$ and $\mathbf{E}$, the sequence $(\mathbf{E}_n)_n$ converges to $\mathbf{E}$ in the local Gromov–Hausdorff sense if for every $r \geq 0$,

$$d_{\mathrm{GH}}\big(B_r(\mathbf{E}_n), B_r(\mathbf{E})\big) \to 0 \qquad \text{as } n \to \infty.$$

This notion of convergence is metrizable (see [20], Section 2.1, for a possible definition of the metric) and turns the space $\mathbb{K}_{\mathrm{bcl}}$ of isometry classes of pointed complete and locally compact length spaces into a Polish space. In passing, we note that a length space $\mathbf{E}$ is complete and locally compact if and only if it is boundedly compact, meaning that all closed balls in $\mathbf{E}$ are compact; see Proposition 2.5.22 of [16].

As discrete planar maps, quadrangulations are clearly not length spaces. Following [20], we may nonetheless interpret a (finite or infinite) quadrangulation Q as a complete and locally compact length space $\mathbf{Q}$. Namely, we replace each edge of Q by an Euclidean segment of length one such that two segments can intersect only at their endpoints, and they do so if and only if the corresponding edges in E share one or two vertices. Equipped with the shortest-path metric, the resulting metric space $\mathbf{Q}$ is then a union of copies of the interval $[0, 1]$, one for each edge of Q. Moreover, with the root vertex of Q as distinguished point, $\mathbf{Q}$ is a (pointed) complete and locally compact length space, and it is easy to see that $d_{\mathrm{GH}}(B_r(Q), B_r(\mathbf{Q})) \le 1$ for every $r \ge 0$.

NOTATION. Given a pointed metric space $\mathbf{E} = (E, d, \rho)$ and $\lambda > 0$, we write $\lambda \cdot \mathbf{E}$ for the dilated (or rescaled) space $(E, \lambda \cdot d, \rho)$. In particular, if $\lambda, \delta > 0$, $\lambda \cdot B_\delta(\mathbf{E}) = B_{\lambda\delta}(\lambda \cdot \mathbf{E})$.

REMARK 2.10. From our observation above, we deduce that our limit results for quadrangulations $Q_n^{\sigma_n}$ in the local Gromov–Hausdorff sense will follow if we show that for each $r \ge 0$, $B_r(a_n^{-1} \cdot Q_n^{\sigma_n})$ converges in distribution in $\mathbb{K}$ toward the ball of radius r in the corresponding limit space. Note that all our limit spaces in the local Gromov–Hausdorff sense, that is, the spaces BP, BHP_θ, IBD_σ and SCRT, are already complete locally compact length spaces. Indeed, real trees are always length spaces, and the metric gluing of length spaces produces again a length space; see the discussion in [16] after Exercice 3.1.13.

We therefore do not have to deal with the more complicated notion of local Gromov–Hausdorff convergence for general (pointed) metric spaces; see [16], Definition 8.1.1.

2.4.3. *Local limits of maps.* Local limits of maps in the sense of Benjamini and Schramm [6] concern the convergence of combinatorial balls. More specifically, given a rooted planar map $\mathfrak{m}$ and $r \ge 0$, write $\mathrm{Ball}_r(\mathfrak{m})$ for the combinatorial of radius r, that is the submap of $\mathfrak{m}$ formed by all the vertices v of $\mathfrak{m}$ with $d_{\mathrm{gr}}(\varrho, v) \le r$, together with the edges of $\mathfrak{m}$ in between such vertices. For two rooted maps $\mathfrak{m}$ and $\mathfrak{m}'$, the local distance between $\mathfrak{m}$ and $\mathfrak{m}'$ is defined as

$$d_{\mathrm{map}}(\mathfrak{m}, \mathfrak{m}') = \left(1 + \sup\{r \ge 0 : \mathrm{Ball}_r(\mathfrak{m}) = \mathrm{Ball}_r(\mathfrak{m}')\}\right)^{-1}.$$

The metric d_{map} induces a topology on the set of all finite quadrangulations (with or without boundary). *Infinite quadrangulations* are the elements in the completion of this space with respect to d_{map} that are not finite quadrangulations (the UIHPQ is a random infinite quadrangulation with an infinite boundary). See [22] for more on this.

3. Main results. We formulate now in a proper way our main results, which cover together with the results of [9, 11] all the convergences listed in the Introduction. The proofs will be given in Section 6, except for the proof of Theorem 3.5, which can be found in the Supplementary Material [3].

3.1. *Scaling limits of quadrangulations with a boundary.* Recall the notation introduced in Section 1.1. All convergences in this section are in law, with respect to the local Gromov–Hausdorff topology. We always consider the limit $n \to \infty$.

THEOREM 3.1. *Assume $\sigma_n \ll \sqrt{n}$ and $\sqrt{\sigma_n} \ll a_n \ll n^{1/4}$. Then*

$$\big(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{BP}.$$

THEOREM 3.2. *Assume $1 \ll \sigma_n \ll \sqrt{n}$ and $a_n \sim (4/9)^{1/4}\sqrt{\sigma_n/\sigma}$ for some $\sigma \in (0, \infty)$. Then*

$$\big(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{IBD}_\sigma.$$

THEOREM 3.3. *Assume $1 \ll \sigma_n \ll n$ and $1 \ll a_n \ll \min\{\sqrt{\sigma_n}, \sqrt{n/\sigma_n}\}$. Then*

$$\big(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{BHP}.$$

THEOREM 3.4. *Assume $\sqrt{n} \ll \sigma_n \ll n$ and $a_n \sim 2\sqrt{\theta n/3\sigma_n}$ for some $\theta \in (0, \infty)$. Then*

$$\big(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{BHP}_\theta.$$

THEOREM 3.5. *Assume $\sigma_n \gg \sqrt{n}$ and $\max\{1, \sqrt{n/\sigma_n}\} \ll a_n \ll \sqrt{\sigma_n}$. Then*

$$\big(V(Q_n^{\sigma_n}), a_n^{-1} d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{SCRT}.$$

When the scaling sequence $(a_n, n \in \mathbb{N})$ satisfies $a_n \gg \max\{\sqrt{\sigma_n}, n^{1/4}\}$, then the limiting space is the trivial one-point metric space. This is a direct consequence of the results in [9], for example.

The Brownian half-plane BHP does also arise as the weak scaling limit of the UIHPQ (similarly, the Brownian plane BP is the scaling limit of the so-called uniform infinite planar quadrangulation UIPQ; see the first part of [20], Theorem 2). The following result was also obtained by Gwynne and Miller in an independent and essentially simultaneous work [24]. Their work includes the convergence of the UIHPQ with a simple boundary toward the BHP, which is left out here.

THEOREM 3.6. $\lambda \cdot \mathsf{UIHPQ} \xrightarrow{\lambda \to 0} \mathsf{BHP}.$

In [4], a similar discrete approximation is given for BHP_θ when $\theta > 0$.

3.2. *Couplings and topology.* For proving Theorem 3.3, we follow a strategy similar to that in Curien and Le Gall [20]. As an intermediate step, we establish a coupling between the Brownian disk BD_σ and the Brownian half-plane BHP_θ.

THEOREM 3.7. *Let $\varepsilon > 0$, $r \geq 0$. Let $\sigma(\cdot) : (0, \infty) \to (0, \infty)$ be a function satisfying $\lim_{T\to\infty} \sigma(T)/T = \theta \in [0, \infty)$ and, in case $\theta = 0$, $\liminf_{T\to\infty} \sigma(T)/\sqrt{T} > 0$. Then there exists $T_0 = T_0(\varepsilon, r, \sigma)$ such that for all $T \geq T_0$, one can construct copies of $\mathsf{BD}_{T,\sigma(T)}$ and BHP_θ on the same probability space such that with probability at least $1 - \varepsilon$, there exist two isometric open subsets O_{BD}, O_{BHP} in these spaces which are both homeomorphic to the closed half-plane $\overline{\mathbb{H}}$ and contain the balls $B_r(\mathsf{BD}_{T,\sigma(T)})$ and $B_r(\mathsf{BHP}_\theta)$, respectively.*

We remark that for the proof of Theorem 3.3, it would be sufficient to show that the balls of radius r around the root in the corresponding spaces are isometric. From the stronger version of the coupling stated above, we can, however, additionally deduce

COROLLARY 3.8. *For every $\theta \geq 0$, the space BHP_θ is a.s. homeomorphic to the closed half-plane $\overline{\mathbb{H}} = \mathbb{R} \times \mathbb{R}_+$.*

Since the Brownian half-plane $\mathsf{BHP} = \mathsf{BHP}_0$ is scale-invariant, that is, $\lambda \cdot \mathsf{BHP} =_d \mathsf{BHP}$ for every $\lambda > 0$, Theorem 3.7 moreover implies that BHP is locally isometric to the disk $\mathsf{BD}_\sigma (= \mathsf{BD}_{1,\sigma})$.

COROLLARY 3.9. *Fix $\sigma \in (0, \infty)$, and let $\varepsilon > 0$. Then one can find $\delta > 0$ and construct on the same probability space copies of BD_σ and BHP such that with probability at least $1 - \varepsilon$, $B_\delta(\mathsf{BHP})$ and $B_\delta(\mathsf{BD}_\sigma)$ are isometric.*

The proof of Corollary 3.9 is immediate from the scaling properties of $\mathsf{BD}_{T,\sigma}$ and BHP, whereas Corollary 3.8 needs an extra argument, which we give in Section 6.2.

REMARK 3.10. The local isometry between BHP and BD_σ together with the fact that BHP is scale-invariant uniquely characterizes the law of BHP in the set of all probability measures on $\mathbb{K}_{\mathrm{bcl}}$. This follows from the argument in the proof of [21], Proposition 3.2, where a similar characterization of the Brownian plane is given.

For establishing Theorem 3.3, we shall also need a coupling between the UIHPQ and $Q_n^{\sigma_n}$ when σ_n grows slower than n.

PROPOSITION 3.11. *Assume $1 \ll \sigma_n \ll n$, and put $\vartheta_n = \min\{\sigma_n, n/\sigma_n\}$. Given any $\varepsilon > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, one* *can construct copies of* $Q_n^{\sigma_n}$ *and* UIHPQ *on the same probability space such that with probability at least* $1-\varepsilon$, *the balls* $B_{\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$ *and* $B_{\delta\sqrt{\vartheta_n}}(\mathsf{UIHPQ})$ *are isometric. Moreover, we have the local convergence*

$$\big(V(Q_n^{\sigma_n}), d_{\mathrm{gr}}, \rho_n\big) \longrightarrow \mathsf{UIHPQ}$$

in distribution for the metric d_{map}, *as* $n \to \infty$.

Note that the above mentioned UIPQ is in turn the weak limit in the sense of d_{map} for uniform quadrangulations *without* a boundary; see Krikun [27].

For proving Theorem 3.2 and determining the topology of the infinite-volume Brownian disk IBD_σ, we couple the Brownian disk $\mathsf{BD}_{T,\sigma}$ for large volumes T with IBD_σ.

THEOREM 3.12. *Fix* $\sigma \in (0,\infty)$, *and let* $\varepsilon > 0$, $r \geq 0$. *There exists* $T_0 = T_0(\varepsilon, r, \sigma)$ *such that for all* $T \geq T_0$, *we can construct copies of* $\mathsf{BD}_{T,\sigma}$ *and* IBD_σ *on the same probability space such that with probability at least* $1-\varepsilon$, *there exist two isometric open subsets* A_{BD}, A_{IBD} *in these spaces which are both homeomorphic to the pointed closed disk* $\overline{\mathbb{D}} \setminus \{0\}$ *and contain the balls* $B_r(\mathsf{BD}_{T,\sigma})$ *and* $B_r(\mathsf{IBD}_\sigma)$, *respectively.*

It will be straightforward to deduce the following.

COROLLARY 3.13. *For each* $\sigma \in (0,\infty)$, *the space* IBD_σ *is a.s. homeomorphic to the pointed closed disk* $\overline{\mathbb{D}} \setminus \{0\}$, *where* $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

In order to prove Theorem 3.2, we finally need a coupling of balls in the quadrangulations $Q_n^{\sigma_n}$ and $Q_{R\sigma_n^2}^{\sigma_n}$ of a radius of order $\sqrt{\sigma_n}$, when $1 \ll \sigma_n \ll \sqrt{n}$ and R is large.

PROPOSITION 3.14. *Assume* $1 \ll \sigma_n \ll \sqrt{n}$. *Given any* $\varepsilon > 0$ *and* $r > 0$, *there exist* $R_0 > 0$ *and* $n_0 \in \mathbb{N}$ *such that for every integer* $R \geq R_0$ *and every* $n \geq n_0$, *one can construct copies of* $Q_n^{\sigma_n}$ *and* $Q_{R\sigma_n^2}^{\sigma_n}$ *on the same probability space such that with probability at least* $1-\varepsilon$, *the balls* $B_{r\sqrt{\sigma_n}}(Q_n^{\sigma_n})$ *and* $B_{r\sqrt{\sigma_n}}(Q_{R\sigma_n^2}^{\sigma_n})$ *are isometric.*

Some of our results involving UIHPQ, BHP and BD_σ are depicted in Figure 3, which should be compared with [20], Figure 1.

3.3. *Limits of the Brownian disk.* Our statements from the last two sections imply various limit results for the Brownian disk $\mathsf{BD}_{T,\sigma(T)}$ when zooming-in around its root. We let $\sigma(\cdot) : (0,\infty) \to (0,\infty)$ be a function of the volume T of the Brownian disk that specifies its perimeter, and we write $\mathcal{X}$ for the distributional limit of $\mathsf{BD}_{T,\sigma(T)}$ in the local Gromov–Hausdorff topology upon letting $T \to \infty$ (if it exists).

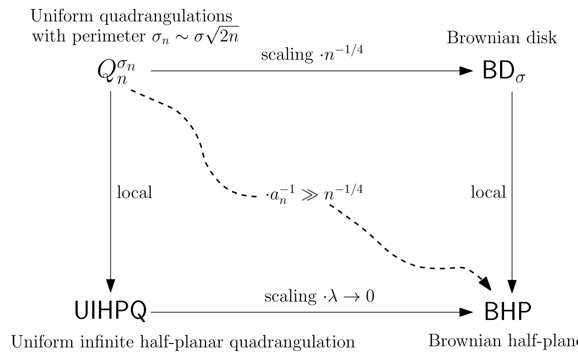

FIG. 3. *Illustration of* [11], *Theorem* 1, *for the regime* $\sigma_n \sim \sigma\sqrt{2n}$, *Theorem* 3.3, *Theorem* 3.6, *Corollary* 3.15 *in the case* $\sigma(T) \equiv \sigma \in (0, \infty)$, *and Proposition* 3.11. *Compare with* [20], *Figure* 1.

COROLLARY 3.15. *We have*

$$\mathcal{X} = \begin{cases} \mathsf{BP} & \text{if } \lim_{T\to\infty} \sigma(T) = 0, \\ \mathsf{IBD}_\varsigma & \text{if } \lim_{T\to\infty} \sigma(T) = \varsigma \in (0, \infty), \\ \mathsf{BHP}_\theta & \text{if } \sigma(T) \to \infty \text{ and } \sigma(T)/T \to \theta \in [0, \infty) \text{ as } T \to \infty, \\ \mathsf{SCRT} & \text{if } \sigma(T)/T \to \infty \text{ as } T \to \infty. \end{cases}$$

Note that the third case includes the case $\sigma(T) = \sqrt{T}$. Then $\theta = 0$, and since by scaling, $T^{1/4} \cdot \mathsf{BD}_1 =_d \mathsf{BD}_{T,\sqrt{T}}$, it follows that BHP is the tangent cone in distribution of any disk $\mathsf{BD}_{A,L}$ for fixed $A, L > 0$. See [16], Section 8.2, for an explanation of this terminology, and compare with [20], Theorem 1, where it is shown that the Brownian plane is the tangent cone of the Brownian map at its root.

For completeness, but without going into details, let us mention that identically to the proof of the first (or last) case of Corollary 3.15, a combination of [11], Theorem 1, and [9], Theorem 4 (or [9], Theorem 4) leads to the convergences

$$\mathsf{BD}_{T,\sigma} \xrightarrow{\sigma\to 0} \mathsf{BM}_T, \qquad \mathsf{BD}_{T,\sigma} \xrightarrow{T\to 0} \mathsf{CRT}_{3\sigma}$$

in law in the sense of the *global* Gromov–Hausdorff topology. The factor 3 in $\mathsf{CRT}_{3\sigma}$ stems from the particular normalization of the Brownian disk.

REMARK/EXERCISE 3.16. We leave it as an exercise to the reader to find the right combination of our (or Bettinelli's; cf. [9]) foregoing results to deduce the following additional results on tangent cones (in distribution, with respect to the local Gromov–Hausdorff topology):

$$\mathsf{CRT}_T \xrightarrow{T\to\infty} \mathsf{SCRT}, \qquad \mathsf{BHP}_\theta \xrightarrow{\theta\to 0} \mathsf{BHP}, \qquad \mathsf{IBD}_\sigma \xrightarrow{\sigma\to\infty} \mathsf{BHP}.$$

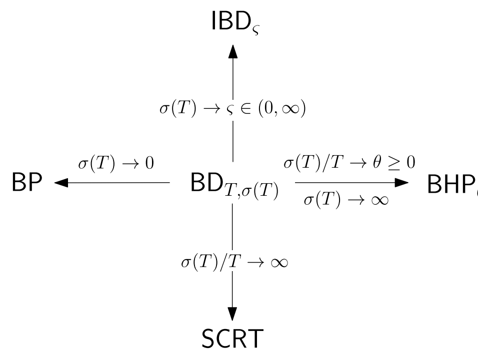

FIG. 4. *Zooming-in around the root of the Brownian disk* $\mathsf{BD}_{T,\sigma(T)}$ *with volume* T *and perimeter* $\sigma(T)$. *The figure shows all possible weak limits in the local Gromov–Hausdorff sense when* $T \to \infty$ (*Corollary* 3.15).

Combining results from the regime $\sigma_n \ll \sqrt{n}$ in the first and from $\sigma_n \gg \sqrt{n}$ in the second case, one may also prove the following scaling results in law:

$$\mathsf{BHP}_\theta \xrightarrow{\theta\to\infty} \mathsf{SCRT}, \qquad \mathsf{IBD}_\sigma \xrightarrow{\sigma\to 0} \mathsf{BP}.$$

In the terminology of [16], Section 8.2, the last two results imply that the SCRT is the asymptotic cone in distribution of BHP_θ for $\theta > 0$, and similarly, BP is the asymptotic cone of IBD_σ.

4. Encoding of quadrangulations with a boundary. We will use a variant of the Cori–Vauquelin–Schaeffer [19, 39] bijection developed by Bouttier, Di Francesco and Guitter [13] to encode quadrangulations with a boundary. More specifically, we will encode planar quadrangulations of size n with a boundary of size 2σ in terms of σ trees with n edges in total, which are attached to a discrete bridge of length σ. We first introduce the encoding objects. Our notation is inspired by [8, 9].

4.1. *Encoding in the finite case.*

4.1.1. *Well-labeled tree, forest and bridge.* A *well-labeled tree* of size $|\tau| = n$ is a pair $(\tau, (\ell(u))_{u\in V(\tau)})$ consisting of a rooted plane tree τ with n edges together with integer labels $(\ell(u))_{u\in V(\tau)}$ attached to the vertices of τ, such that the root has label 0, and $|\ell(u) - \ell(v)| \leq 1$ whenever u and v are neighbors.

A *well-labeled forest* with σ trees and n tree edges is a collection $\mathfrak{f} = (\tau_0, \dots, \tau_{\sigma-1})$ of σ trees with n edges in total, together with a labeling of vertices $\mathfrak{l} : \bigcup_{i=0}^{\sigma-1} V(\tau_i) \to \mathbb{Z}$, which has the property that for each $i = 0, \dots, \sigma - 1$, the tree τ_i together with the restriction $\mathfrak{l} \restriction V(\tau_i)$ forms a well-labeled tree.

The vertex set of $\mathfrak{f}$ is $V(\mathfrak{f}) = \bigcup_{i=0}^{\sigma-1} V(\tau_i)$. Note that $|V(\mathfrak{f})| = n + \sigma$. The size of $\mathfrak{f}$ is given by $|\mathfrak{f}| = n$, that is, its number of edges. We write $(0), \dots, (\sigma - 1)$

for the root vertices of $\tau_0, \dots, \tau_{\sigma-1}$. If u is a vertex of a tree of $\mathfrak{f}$, $\mathfrak{r}(u)$ denotes the root of this tree. In particular, the vertex set of the jth tree of $\mathfrak{f}$ is the set $\{u \in V(\mathfrak{f}) : \mathfrak{r}(u) = (j-1)\}$, $j = 1, \dots, \sigma$. We write $t(\mathfrak{f}) = \sigma$ for the number of trees of $\mathfrak{f}$. We will often identify the root vertices with the integers $0, \dots, \sigma - 1$ and consequently regard $\mathfrak{r}(u)$ as a number.

We call the pair $(\mathfrak{f}, \mathfrak{l})$ a *well-labeled forest* and denote by

$$\mathfrak{F}_\sigma^n = \{(\mathfrak{f}, \mathfrak{l}) : t(\mathfrak{f}) = \sigma, |\mathfrak{f}| = n\}$$

the set of all well-labeled forests of size n with σ trees.

A *bridge* of length $\sigma \geq 1$ is a sequence of numbers $(\mathsf{b}(0), \mathsf{b}(1), \dots, \mathsf{b}(\sigma))$ with $\mathsf{b}(0) = 0$ and such that $\mathsf{b}(i+1) - \mathsf{b}(i) \in \mathbb{N}_0 \cup \{-1\}$ for $i = 0, \dots, \sigma - 1$, and $\mathsf{b}(\sigma) \leq 0$.

By linear interpolation between integer values, we will view $\mathsf{b} : [0, \sigma] \to \mathbb{R}$ as a continuous function and write $\mathfrak{B}_\sigma \subset \mathcal{C}([0, \sigma], \mathbb{R})$ for the set of all possible bridges of length σ.

The terminal value $\mathsf{b}(\sigma)$ of a bridge has a special interpretation: It keeps the information where to find the root in the quadrangulation associated to a triplet $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$; see Section 4.3 below.

4.1.2. *Contour pair and label function.* Consider a well-labeled forest $(\mathfrak{f}, \mathfrak{l})$ of size n with σ trees. In order to define its contour pair and label function, it is convenient to represent $(\mathfrak{f}, \mathfrak{l})$ in the plane, as depicted in Figure 5. We add $\sigma - 1$ edges which link the root vertices $(0), \dots, (\sigma - 1)$, such that vertex $(i-1)$ gets connected to (i) for $i = 1, \dots, \sigma - 1$, plus an extra vertex (σ) and an extra edge linking $(\sigma - 1)$ to (σ). We extend $\mathfrak{l}$ to (σ) by setting $\mathfrak{l}((\sigma)) = 0$. We refer to the segment connecting the roots of $\mathfrak{f}$ and the extra vertex (σ) as the *floor* of $\mathfrak{f}$.

The *facial sequence* $\mathfrak{f}(0), \dots, \mathfrak{f}(2n+\sigma)$ of $\mathfrak{f}$ is the sequence of vertices obtained from exploring (the embedding of) $\mathfrak{f}$ in the contour order, starting from vertex (0). In other words, $\mathfrak{f}(0), \dots, \mathfrak{f}(2n + \sigma - 1)$ is given by the sequence of vertices of the discrete contour paths of the trees $\tau_0, \dots, \tau_{\sigma-1}$, and the sequence terminates

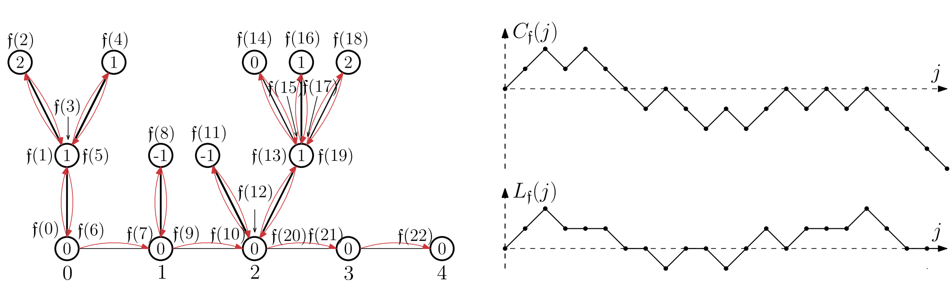

FIG. 5. *On the left: A proper representation of a finite well-labeled forest* $(\mathfrak{f}, \mathfrak{l})$ *of size* 13 *with* 4 *trees, together with its facial sequence. The rightmost vertex indexed by* 4 *is the added extra vertex. On the right: Its contour pair.*

with value $\mathfrak{f}(2n+\sigma)=(\sigma)$; see, for example, [31], Section 2, for more on contour paths.

Given a well-labeled forest $(\mathfrak{f},\mathfrak{l})$, we define its *contour pair* $(C_{\mathfrak{f}}, L_{\mathfrak{f}})$ by

$$C_{\mathfrak{f}}(j)=d_{\mathfrak{f}}\big(\mathfrak{f}(j),(\sigma)\big)-\sigma, \qquad L_{\mathfrak{f}}(j)=\mathfrak{l}\big(\mathfrak{f}(j)\big), \qquad j=0,\dots,2n+\sigma.$$

Here, $d_{\mathfrak{f}}$ denotes the graph distance on the representation of $\mathfrak{f}$ in the plane.

We call $C_{\mathfrak{f}}$ the *contour function* of $\mathfrak{f}$, since it is obtained from concatenating the contour paths of the trees $\tau_0,\dots,\tau_{\sigma-1}$, with an additional -1 step after a tree has been visited. Note that $L_{\mathfrak{f}}(\mathfrak{f}(j))=0$ if $\mathfrak{f}(j)$ lies on the floor of $\mathfrak{f}$; see again Figure 5 for an illustration.

Now consider additionally a bridge $\mathsf{b}\in\mathfrak{B}_\sigma$. Put $\underline{C}_{\mathfrak{f}}(j)=\min_{[0,j]}C_{\mathfrak{f}}$. The function

$$\mathfrak{L}_{\mathfrak{f}}(j)=L_{\mathfrak{f}}(j)+\mathsf{b}\big(-\underline{C}_{\mathfrak{f}}(j)\big), \qquad j=0,\dots,2n+\sigma,$$

is called the *label function* associated to $((\mathfrak{f},\mathfrak{l}),\mathsf{b})$. The label function plays an important role in measuring distances in the quadrangulation associated through the Bouttier–Di Francesco–Guitter bijection; see Section 4.5.1.

By linear interpolation between integers, we extend all three functions $C_{\mathfrak{f}}$, $L_{\mathfrak{f}}$ and $\mathfrak{L}_{\mathfrak{f}}$ to continuous real-valued functions on $[0,2n+\sigma]$.

4.2. *Encoding in the infinite case*. We next introduce the infinite analogs of the objects from the previous section. They will encode certain infinite quadrangulations with an infinite boundary.

4.2.1. *Well-labeled infinite forest and infinite bridge*. A *well-labeled infinite forest* is an infinite collection $\mathfrak{f}=(\tau_i, i\in\mathbb{Z})$ of finite rooted plane trees, together with a labeling of vertices $\mathfrak{l}:\bigcup_{i\in\mathbb{Z}}V(\tau_i)\to\mathbb{Z}$ such that for each $i\in\mathbb{Z}$, τ_i together with the restriction of $\mathfrak{l}$ to $V(\tau_i)$ forms a well-labeled tree.

We write again (k) for the root vertex of τ_k and often identify (k) with $k\in\mathbb{Z}$. The set of all well-labeled infinite forests $(\mathfrak{f},\mathfrak{l})$ will be denoted by $\mathfrak{F}_\infty$.

An *infinite bridge* is a sequence of numbers $\mathsf{b}=(\mathsf{b}(i), i\in\mathbb{Z}\cup\{\partial\})$ with $\mathsf{b}(0)=0$, $\mathsf{b}(i+1)-\mathsf{b}(i)\in\mathbb{N}_0\cup\{-1\}$ for all $i\in\mathbb{Z}$ and $\mathsf{b}(\partial)\in\{\mathsf{b}(-1)-1,\dots,0\}$. Note that $\mathsf{b}(-1)\le 1$.

The extra value $\mathsf{b}(\partial)$ will keep track of the position of the root in the quadrangulation. Often, we consider only the values $\mathsf{b}(i)$, $i\in\mathbb{Z}$, and then view b as a continuous function from $\mathbb{R}$ to $\mathbb{R}$, by linear interpolation between integer values. We write $\mathfrak{B}_\infty$ for the set of all infinite bridges b which have the property that $\inf_{i\in\mathbb{N}}\mathsf{b}(i)=-\infty$, and $\inf_{i\in\mathbb{N}}\mathsf{b}(-i)=-\infty$.

4.2.2. *Contour pair and label function in the infinite case*. We consider a well-labeled infinite forest $(\mathfrak{f},\mathfrak{l})\in\mathfrak{F}_\infty$. Again, we view $\mathfrak{f}$ as a graph properly embedded in the plane (Figure 6): We identify the set of roots of the trees of $\mathfrak{f}$ with $\mathbb{Z}$ and

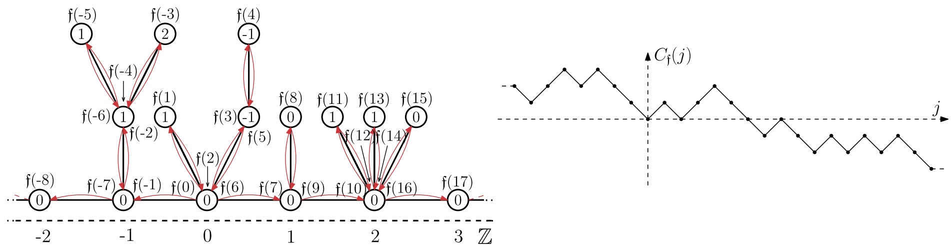

FIG. 6. *On the left*: *A proper representation of a well-labeled infinite forest* $(\mathfrak{f}, \mathfrak{l})$, *together with its facial sequence. On the right*: *Its contour function.*

connect neighboring roots by an edge. We obtain what we call the *floor* of $\mathfrak{f}$. The trees τ_i of $\mathfrak{f}$ are drawn in the upper half-plane and attached to the floor.

The *facial sequence* $(\mathfrak{f}(i), i \in \mathbb{Z})$ of $\mathfrak{f}$ is defined as follows: $(\mathfrak{f}(0), \mathfrak{f}(1), \ldots)$ is the sequence of vertices of the contour paths of the trees $\tau_i, i \in \mathbb{N}_0$, in the contour order, starting from the root of the tree τ_0, and $(\mathfrak{f}(-1), \mathfrak{f}(-2), \ldots)$ is given by the sequence of vertices of the contour paths $\tau_{-1}, \tau_{-2}, \ldots,$ in the *counterclockwise* order, starting from the root of the tree τ_{-1}.

In analogy to the finite case, given a well-labeled infinite forest $(\mathfrak{f}, \mathfrak{l})$, its *contour pair* $(C_{\mathfrak{f}}, L_{\mathfrak{f}})$ is the pair of functions defined *via*

$$C_{\mathfrak{f}}(j) = d_{\mathfrak{f}}\big(\mathfrak{f}(j), \mathfrak{r}\big(\mathfrak{f}(j)\big)\big) - \mathfrak{r}\big(\mathfrak{f}(j)\big), \qquad L_{\mathfrak{f}}(j) = \mathfrak{l}\big(\mathfrak{f}(j)\big), \qquad j \in \mathbb{Z},$$

where $d_{\mathfrak{f}}$ is the graph distance on the embedding of $\mathfrak{f}$, and $\mathfrak{r}(\mathfrak{f}(j))$ denotes the root of the tree $\mathfrak{f}(j)$ belongs to. Be aware of the small abuse of notation: In the expression for $C_{\mathfrak{f}}$, $\mathfrak{r}(\mathfrak{f}(j))$ is first viewed as a vertex and then as an integer. As for a finite forest, we call $C_{\mathfrak{f}}$ the *contour function* of $\mathfrak{f}$.

If additionally $\mathsf{b} \in \mathfrak{B}_{\infty}$, we define the *label function* associated to $((\mathfrak{f}, \mathfrak{l}), \mathsf{b})$ by

$$\mathfrak{L}_{\mathfrak{f}}(j) = L_{\mathfrak{f}}(j) + \mathsf{b}\big(-\underline{C}_{\mathfrak{f}}(j)\big), \qquad j \in \mathbb{Z}, \qquad \mathfrak{L}_{\mathfrak{f}}(\partial) = \mathsf{b}(\partial),$$

where $\underline{C}_{\mathfrak{f}}(j) = \inf_{(-\infty, j]} C_{\mathfrak{f}}$ for $j < 0$ and $\underline{C}_{\mathfrak{f}}(j) = \min_{[0,j]} C_{\mathfrak{f}}$ for $j \geq 0$, as above.

Again by linear interpolation between integers, we view $C_{\mathfrak{f}}$, $L_{\mathfrak{f}}$ and $\mathfrak{L}_{\mathfrak{f}}$ as continuous functions on $\mathbb{R}$.

4.3. *Bouttier–Di Francesco–Guitter bijection.* Recall that a rooted quadrangulation with a boundary comes with a distinguished edge along the boundary, the root edge, whose origin is the root vertex. We write $\mathcal{Q}_n^{\sigma}$ for the set of all rooted quadrangulations with n inner faces and a boundary of size 2σ.

A *pointed quadrangulation with a boundary* is a pair $(\mathsf{q}, v^{\bullet})$, where q is a rooted quadrangulation with a boundary and $v^{\bullet} \in V(\mathsf{q})$ is a distinguished vertex. The set of all rooted pointed quadrangulations with n internal faces and 2σ boundary edges is denoted by

$$\mathcal{Q}_{n,\sigma}^{\bullet} = \big\{(\mathsf{q}, v^{\bullet}) : \mathsf{q} \in \mathcal{Q}_n^{\sigma}, v^{\bullet} \in V(\mathsf{q})\big\}.$$

4.3.1. *The finite case.* The Bouttier–Di Francesco–Guitter bijection [13] provides us with a bijection

$$\Phi_n : \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma \longrightarrow \mathcal{Q}_{n,\sigma}^\bullet.$$

We shall here content ourselves with the description of the mapping from the encoding objects to the quadrangulations. We follow largely the presentation in [9], where also a description of the reverse direction can be found.

In this regard, let $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$. Out of this triplet, we will now construct a rooted pointed quadrangulation $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}_{n,\sigma}^\bullet$. Recall the facial sequence $\mathfrak{f}(0), \dots, \mathfrak{f}(2n+\sigma)$ of $\mathfrak{f}$ obtained from exploring the trees of $\mathfrak{f}$ in the contour order, as well as the associated label function $\mathfrak{L}_\mathfrak{f}$. We view $\mathfrak{f}$ as embedded in the plane (as explained above) and add an additional vertex $v^\bullet$ inside the only face of $\mathfrak{f}$, with label $\mathfrak{L}_\mathfrak{f}(v^\bullet) = -\infty$.

The vertex set of $\mathfrak{q}$ is given by $V(\mathfrak{f}) \cup \{v^\bullet\}$. Note that by definition, the additional vertex (σ) which forms part of the embedding of $\mathfrak{f}$ is not an element of $V(\mathfrak{f})$. In order to specify the edges between the vertices of $\mathfrak{q}$, we define for $i = 0, \dots, 2n + \sigma - 1$ the *successor* $\mathsf{succ}(i) \in \{0, \dots, 2n+\sigma-1\} \cup \{\infty\}$ of i to be the first number k in the list $(i+1, \dots, 2n+\sigma-1, 0, \dots, i-1)$ with the property that $\mathfrak{L}_\mathfrak{f}(k) = \mathfrak{L}_\mathfrak{f}(i) - 1$, with $\mathrm{succ}(i) = \infty$ if there is no such number. Letting $\mathfrak{f}(\infty) = v^\bullet$, we now follow the facial sequence of $\mathfrak{f}$ and draw for every $i = 0, \dots, 2n+\sigma-1$ an arc between $\mathfrak{f}(i)$ and $\mathfrak{f}(\mathrm{succ}(i))$, in such a way that it neither crosses arcs that were previously drawn, nor edges of the embedding of $\mathfrak{f}$. Since any vertex of $\mathfrak{f}$ which is not a leaf is visited at least twice in the contour exploration, there can be several arcs connecting $\mathfrak{f}(i)$ and $\mathfrak{f}(\mathrm{succ}(i))$. By a small abuse of language, we therefore speak of the arc connecting i to $\mathrm{succ}(i)$ and write

$$i \curvearrowright \mathrm{succ}(i) \quad \text{or} \quad i \curvearrowleft \mathrm{succ}(i)$$

for the oriented arc from i toward $\mathrm{succ}(i)$ or from $\mathrm{succ}(i)$ toward i, respectively.

The arcs between the vertices $V(\mathfrak{f}) \cup \{v^\bullet\}$ form the edges of $\mathfrak{q}$, and it remains to specify the root edge of $\mathfrak{q}$: Denoting by succ^k the kth functional power of the function succ (with $\mathrm{succ}(\infty) = \infty$), the root vertex is given by $\mathfrak{f}(\mathrm{succ}^{-\mathsf{b}(\sigma)}(0))$, and the root edge is in case $\mathsf{b}(\sigma) > \mathsf{b}(\sigma-1) - 1$ given by $\mathrm{succ}^{-\mathsf{b}(\sigma)}(0) \curvearrowright \mathrm{succ}^{-\mathsf{b}(\sigma)+1}(0)$, and in case $\mathsf{b}(\sigma) = \mathsf{b}(\sigma-1) - 1$ by $2n+\sigma-1 \curvearrowleft \mathrm{succ}(2n+\sigma-1)$. Note that in the second case, we have indeed $\mathfrak{f}(\mathrm{succ}(2n+\sigma-1)) = \mathfrak{f}(\mathrm{succ}^{-\mathsf{b}(\sigma)}(0))$, that is, $\mathfrak{f}(\mathrm{succ}(2n+\sigma-1))$ is the root vertex; see Figure 7.

4.3.2. *The infinite case.* Let $\mathcal{Q}$ denote the completion of the space of all rooted finite quadrangulations with a boundary with respect to d_{map}. We extend Φ_n to a mapping

$$\Phi : \Big(\bigcup_{n,\sigma\in\mathbb{N}} \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma\Big) \cup (\mathfrak{F}_\infty \times \mathfrak{B}_\infty) \longrightarrow \mathcal{Q}$$

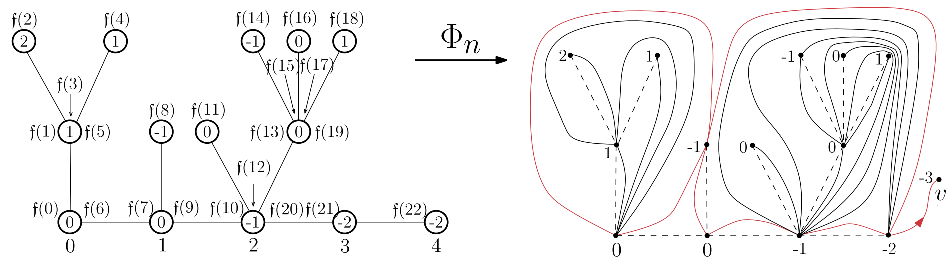

FIG. 7. *The Bouttier–Di Francesco–Guitter bijection* Φ_n *applied to an element* $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$. *The forest* $\mathfrak{f}$ *is the same as in Figure 5, but the labels are shifted by the values of the bridge* b. *The (nonsimple) boundary of the associated quadrangulation on the left is represented in red. Note that the extra vertex* $v^\bullet$ *is in this example a boundary vertex. The rightmost vertex* $\mathfrak{f}(22) = (4)$ *on the left is not a vertex of the quadrangulation. Its label* -2 *captures the information where to find the root edge, which is indicated by an arrow.*

as follows. For elements $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$, we let $\Phi((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) = \Phi_n((\mathfrak{f}, \mathfrak{l}), \mathsf{b})$, where we view the latter as an element in $\mathcal{Q}_n^\sigma$, by simply forgetting its distinguished vertex.

Now let $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\infty \times \mathfrak{B}_\infty$. For $i \in \mathbb{Z}$, we define the *successor* $\mathrm{succ}_\infty(i)$ to be the smallest number k greater than i such that $\mathfrak{L}_\mathfrak{f}(k) = \mathfrak{L}_\mathfrak{f}(i) - 1$. Note that since $\inf_{i \in \mathbb{N}} \mathsf{b}(i) = -\infty$, the definition make sense. We consider a proper embedding of $\mathfrak{f}$ in the plane as described above and draw an arc between $\mathfrak{f}(i)$ and $\mathfrak{f}(\mathrm{succ}_\infty(i))$, for any $i \in \mathbb{Z}$, as indicated by Figure 8. Again we can do this in a way such that arcs do not cross. The vertex set of $\Phi((\mathfrak{f}, \mathfrak{l}), \mathsf{b})$ is given by $V(\mathfrak{f})$, and the edges are the arcs we constructed. Finally, we follow a rooting convention which is analogous to the finite case (we adapt the notion $i \curvearrowright \mathrm{succ}_\infty(i)$ in the obvious way): The root

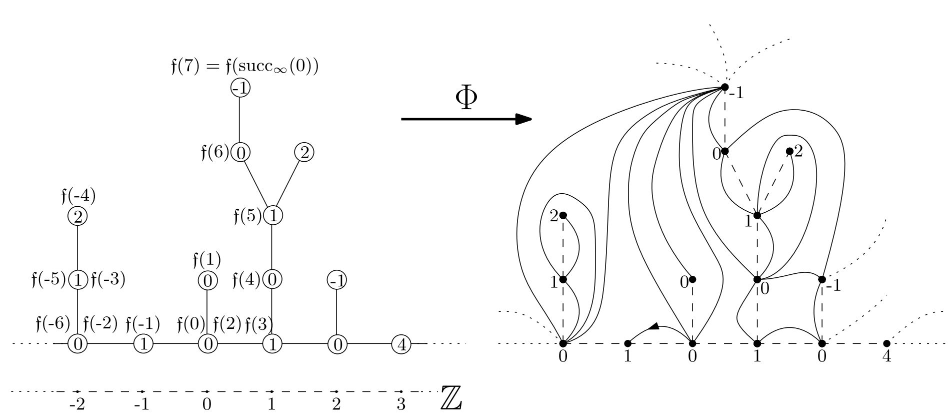

FIG. 8. *The Bouttier–Di Francesco–Guitter mapping applied to an element* $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\infty \times \mathfrak{B}_\infty$. *The successor of* 0 *is* 7, *which is also the successor of* $-6, -2, 1, 2, 4, 6$. *The vertex labels are given by* $\mathfrak{L}_\mathfrak{f}$, *as in Figure 7. The root edge of the map is the oriented arc* $-1 \curvearrowleft \mathrm{succ}_\infty(-1)$ *indicated by an arrow.*

vertex is given by $\mathfrak{f}(\mathrm{succ}_\infty^{-\mathsf{b}(\partial)}(0))$, and the root edge is in case $\mathsf{b}(\partial) > \mathsf{b}(-1) - 1$ given by $\mathrm{succ}_\infty^{-\mathsf{b}(\partial)}(0) \curvearrowright \mathrm{succ}_\infty^{-\mathsf{b}(\partial)+1}(0)$, and in case $\mathsf{b}(\partial) = \mathsf{b}(-1) - 1$ by $-1 \curvearrowleft \mathrm{succ}_\infty(-1)$.

REMARK 4.1. Notice that a triplet $((\mathfrak{f}, \mathfrak{l}), \mathsf{b})$ in $\mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$ or in $\mathfrak{F}_\infty \times \mathfrak{B}_\infty$ is uniquely determined by its associated contour and label functions $(C_\mathfrak{f}, \mathfrak{L}_\mathfrak{f})$. In particular, it makes sense to speak of the quadrangulation associated to $(C_\mathfrak{f}, \mathfrak{L}_\mathfrak{f})$. The distinguished vertex $v^\bullet$ in the finite case will play no particular role in our statements, since we view quadrangulations as metric spaces pointed at their root vertices.

4.4. *Construction of the* UIHPQ. We first introduce an $\mathfrak{F}_\infty$-valued random element $(\mathfrak{f}_\infty, \mathfrak{l}_\infty)$ together with a $\mathfrak{B}_\infty$-valued random element b_∞, which will encode the UIHPQ.

4.4.1. *Uniformly labeled critical infinite forest*. Let τ be a finite random plane tree. Conditionally on τ, we assign a sequence of i.i.d. random variables with the uniform distribution on $\{-1, 0, 1\}$ to the edges of τ. The label $\ell(u)$ of a vertex u of τ is defined to be the sum of the random variables along the edges of the (unique) path from the root to u. Such a random labeling $\ell : V(\tau) \to \mathbb{Z}$ is referred to as a *uniform labeling*. If the tree τ is a Galton–Watson tree with a geometric offspring distribution of parameter $1/2$, we say that τ is a *critical geometric Galton–Watson tree*. If ℓ is a uniform labeling of τ, we refer to the pair $(\tau, (\ell(u))_{u \in V(\tau)})$ as a *uniformly labeled critical geometric Galton–Watson tree*.

A *uniformly labeled critical infinite forest* is a random element $(\mathfrak{f}_\infty, \mathfrak{l}_\infty)$ taking values in $\mathfrak{F}_\infty$ such that the pairs $(\tau_i, \mathfrak{l}_\infty \restriction V(\tau_i))$, $i \in \mathbb{Z}$, are independent uniformly labeled critical geometric Galton–Watson trees.

4.4.2. *Uniform infinite bridge*. Let $\mathsf{b}_\infty = (\mathsf{b}_\infty(i), i \in \mathbb{Z})$ be a two-sided random walk starting from 0 at time 0, that is, $\mathsf{b}_\infty(0) = 0$, which has independent increments given by

$$\mathbb{P}\big(\mathsf{b}_\infty(i) - \mathsf{b}_\infty(i-1) = k\big) = 2^{-k-2}, \qquad k \in \mathbb{N}_0 \cup \{-1\}, \text{ for } i \in \mathbb{Z} \setminus \{0\},$$

and

$$\mathbb{P}\big(-\mathsf{b}_\infty(-1) = k\big) = (k+2)2^{-(k+3)}, \qquad k \in \mathbb{N}_0 \cup \{-1\}.$$

Note that $-\mathsf{b}_\infty(-1)$ has same law as $G + G' - 1$ for G and G' two independent geometric random variables of parameter $1/2$. This follows from the well-known fact that $G + G' + 1$ is distributed as a size-biased geometric random variable. We refer to Section 4.5.2 for more explanations. Next, given $\mathsf{b}_\infty(-1)$, we let $\mathsf{b}_\infty(\partial)$ be a uniformly distributed random variable in $\{\mathsf{b}_\infty(-1) - 1, \dots, 0\}$, independent of everything else.

We call the random element $\mathsf{b}_\infty = (\mathsf{b}_\infty(i), i \in \mathbb{Z} \cup \{\partial\})$ with values in $\mathfrak{B}_\infty$ the *uniform infinite bridge*.

We review now the construction of the UIHPQ given in [23]. Note that there the encoding is defined in a slightly different (but equivalent) manner, and the root edge is oriented in the opposite direction. The following definition is justified by Proposition 3.11.

DEFINITION 4.2. Let $(\mathfrak{f}_\infty, \mathfrak{l}_\infty)$ be a uniformly labeled critical infinite forest, and let b_∞ be a uniform infinite bridge independent of $(\mathfrak{f}_\infty, \mathfrak{l}_\infty)$. The uniform infinite half-planar quadrangulation UIHPQ is the (rooted) random infinite quadrangulation $Q_\infty^\infty = (V(Q_\infty^\infty), d_{\mathrm{gr}}, \rho)$ with an infinite boundary obtained from applying the Bouttier–Di Francesco–Guitter mapping Φ to $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$.

In [23], it was shown that in the sense of d_{map}, there are the weak convergences $Q_n^{\sigma_n} \to Q_\infty^\sigma$ as $n \to \infty$, and $Q_\infty^\sigma \to Q_\infty^\infty$ as $\sigma \to \infty$, where Q_∞^σ is the so-called (rooted) uniform infinite planar quadrangulation with a boundary of perimeter 2σ. We also point at the recent work [17], where a construction of the UIHPQ with a positivity constraint on labels is given, similar to the Chassaing–Durhuus construction [18] of the UIPQ.

REMARK 4.3. We stress that while we use the notation $(\mathfrak{f}, \mathfrak{l})$ for both a finite or infinite (deterministic) well-labeled forest, and similarly, b represents a finite or infinite bridge, $(\mathfrak{f}_\infty, \mathfrak{l}_\infty) \in \mathfrak{F}_\infty$ and $\mathsf{b}_\infty \in \mathfrak{B}_\infty$ will always stand for *random* elements with the particular law just described. We will implicitly assume that b_∞ is independent of $(\mathfrak{f}_\infty, \mathfrak{l}_\infty)$. Similarly, for given σ_n, $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ will denote a random element with the uniform distribution on $\mathfrak{F}_n^{\sigma_n} \times \mathfrak{B}_{\sigma_n}$; see Section 4.5.4.

4.5. *Some ramifications*. We gather here some consequences and remarks which we will tacitly use in the following. We begin with some observations concerning the Bouttier–Di Francesco–Guitter bijection.

4.5.1. *Distances*. Let $(\mathsf{q}, v^\bullet) \in \mathcal{Q}_{n,\sigma}^\bullet$ be a (rooted) pointed quadrangulation of size n with a boundary of size 2σ. Then $(\mathsf{q}, v^\bullet)$ corresponds to a pair $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\sigma^n \times \mathfrak{B}_\sigma$ *via* the Bouttier–Di Francesco–Guitter bijection, and the sets $V(\mathsf{q}) \setminus \{v^\bullet\}$ and $V(\mathfrak{f})$ are identified through this bijection. Recall that the label function $\mathfrak{L} = \mathfrak{L}_f$ represents the labels shifted tree by tree according to the values of the bridge b. By a slight abuse of notation, we will view $\mathfrak{L}$ also as a function on $V(\mathsf{q}) \setminus \{v^\bullet\}$ (or $V(\mathfrak{f})$): If $v \in V(\mathsf{q}) \setminus \{v^\bullet\}$, there is at least one $i \in \{0, \dots, 2n + \sigma - 1\}$ such that v is visited in the ith step of the contour exploration, and we let $\mathfrak{L}(v) = \mathfrak{L}(i)$. Note that this definition makes sense, since $\mathfrak{L}(i) = \mathfrak{L}(j)$ if $\mathfrak{f}(i) = \mathfrak{f}(j)$.

Write $d_{\mathbf{q}}$ for the graph distance on $\mathbf{q}$. From the description of the bijection above, we deduce that

$$d_{\mathbf{q}}(u, v^\bullet) = \mathfrak{L}(u) - \min \mathfrak{L} + 1. \tag{4.1}$$

Moreover, if v_0 is the root vertex of $\mathfrak{q}$, we know that its distance to vertex $\mathfrak{f}(0) = (0)$ is

$$d_{\mathbf{q}}(v_0, (0)) = -\mathsf{b}(\sigma). \tag{4.2}$$

In general, there is no simple formula for distances in $\mathfrak{q}$. However, as we explain next, there exist lower and upper bounds in terms of $\mathfrak{L}$.

We first discuss a lower bound. If $u, v \in V(\mathfrak{f})$ are vertices of the same tree τ of $\mathfrak{f}$, that is, $\mathfrak{r}(u) = \mathfrak{r}(v)$, we let $[\![u, v]\!]$ be the vertex set of the unique injective path in τ connecting u to v. If (i), (j) are two tree roots of $\mathfrak{f}$ with $i < j$, we let $[\![(i), (j)]\!]$ denote the sequence of root vertices $(i), (i+1), \dots, (j)$. For the remaining cases, if $\mathfrak{r}(u) < \mathfrak{r}(v)$, we put

$$[\![u, v]\!] = [\![u, \mathfrak{r}(u)]\!] \cup [\![\mathfrak{r}(u), \mathfrak{r}(v)]\!] \cup [\![v, \mathfrak{r}(v)]\!],$$

whereas if $\mathfrak{r}(v) < \mathfrak{r}(u)$, we let

$$[\![u, v]\!] = [\![u, \mathfrak{r}(u)]\!] \cup [\![\mathfrak{r}(u), (\sigma - 1)]\!] \cup [\![(0), \mathfrak{r}(v)]\!] \cup [\![v, \mathfrak{r}(v)]\!].$$

Now let $u, v \in V(\mathfrak{q}) \setminus \{v^\bullet\}$. The so-called *cactus bound* states that

$$d_{\mathbf{q}}(u, v) \geq \mathfrak{L}(u) + \mathfrak{L}(v) - 2\max\Big\{\min_{[\![u,v]\!]} \mathfrak{L}, \min_{[\![v,u]\!]} \mathfrak{L}\Big\}. \tag{4.3}$$

See [35], Proposition 2.3.8, for a proof in a slightly different context, which is readily adapted to our setting. Since vertex (0) has label $\mathfrak{L}(0) = 0$ and $\mathfrak{L}$ coincides with the values of the bridge along the floor of $\mathfrak{f}$, the distance $d_{\mathbf{q}}((0), u)$ for $u \in V(\mathfrak{q}) \setminus \{v^\bullet\}$ is lower bounded by

$$d_{\mathbf{q}}((0), u) \geq -\max\Big\{\min_{[0,\mathfrak{r}(u)]} \mathsf{b}, \min_{[\mathfrak{r}(u),\sigma-1]} \mathsf{b}\Big\}. \tag{4.4}$$

For an upper bound of $d_{\mathbf{q}}(u, v)$ when $u, v \in V(\mathfrak{q}) \setminus \{v^\bullet\}$, choose $i, j \in \{0, \dots, 2n + \sigma - 1\}$ such that $\mathfrak{f}(i) = u$ and $\mathfrak{f}(j) = v$. Define

$$\overrightarrow{[i, j]} = \begin{cases} \{i, \dots, j\} & \text{if } i \leq j, \\ \{i, \dots, 2n + \sigma - 1\} \cup \{0, \dots, j\} & \text{if } i > j. \end{cases}$$

Then there is the upper bound (see [33], Lemma 3, for a proof)

$$d_{\mathbf{q}}(u, v) \leq \mathfrak{L}(u) + \mathfrak{L}(v) - 2\max\Big\{\min_{\overrightarrow{[i,j]}} \mathfrak{L}(\mathfrak{f}), \min_{\overrightarrow{[j,i]}} \mathfrak{L}(\mathfrak{f})\Big\} + 2. \tag{4.5}$$

Bounds similar to (4.3), (4.4) and (4.5) can be formulated for infinite quadrangulations $\mathfrak{q}_\infty$ constructed from triplets $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}_\infty \times \mathfrak{B}_\infty$. For example, if $u, v \in V(\mathfrak{f})$ with $\mathfrak{r}(u) \leq \mathfrak{r}(v)$, the cactus bound (4.3) reads

$$d_{\mathbf{q}_\infty}(u, v) \geq \mathfrak{L}(u) + \mathfrak{L}(v) - 2\min_{[\![u,v]\!]} \mathfrak{L}. \tag{4.6}$$

4.5.2. *Bridges*. We will need some properties of elements in $\mathfrak{B}_\sigma$. First, as it is shown in [9], Lemma 6, by identifying a bridge $(\mathsf{b}(i), 0 \le i \le \sigma) \in \mathfrak{B}_\sigma$ with the sequence

$$(\underbrace{+1, \dots, +1}_{\substack{\mathsf{b}(0)-\mathsf{b}(\sigma)\\ \text{times}}}, -1, \underbrace{+1, \dots, +1}_{\substack{\mathsf{b}(1)-\mathsf{b}(0)+1\\ \text{times}}}, -1, \underbrace{+1, \dots, +1}_{\substack{\mathsf{b}(2)-\mathsf{b}(1)+1\\ \text{times}}}, \dots, -1, \underbrace{+1, \dots, +1}_{\substack{\mathsf{b}(\sigma)-\mathsf{b}(\sigma-1)+1\\ \text{times}}}), \tag{4.7}$$

one obtains a one-to-one correspondence between $\mathfrak{B}_\sigma$ and the set of sequences in $\{-1, +1\}^{2\sigma}$ counting exactly σ times the number -1. As a consequence, $|\mathfrak{B}_\sigma| = \binom{2\sigma}{\sigma}$.

It is helpful to adopt the following point of view. Imagine that we mark σ points on the discrete circle $\mathbb{Z}/2\sigma\mathbb{Z}$ uniformly at random. Marked points obtain label -1, unmarked points label $+1$. Now choose uniformly at random one of the 2σ circle points as the origin. By walking around the circle in the clockwise order starting from the chosen origin, one observes a sequence of consecutive $+1$ and -1, which is distributed as (4.7) when b is chosen uniformly at random in $\mathfrak{B}_\sigma$. In particular, $(\mathsf{b}(\sigma) - \mathsf{b}(\sigma - 1) + 1) + (\mathsf{b}(0) - \mathsf{b}(\sigma) + 1) = -\mathsf{b}(\sigma - 1) + 2$ has the law of a size-biased pick among all σ consecutive segments of the form $(+1, +1, \dots, +1, -1)$. When σ tends to infinity, it is readily seen that $-\mathsf{b}(\sigma - 1)$ converges in distribution to $G + G' - 1$, where G and G' are two independent geometric random variables of parameter $1/2$. This explains the particular law of the increment $-\mathsf{b}_\infty(-1)$ of a uniform infinite bridge b_∞ that forms part of the encoding of the UIHPQ.

Next, let $(X_i, i \in \mathbb{N})$ be a sequence of i.i.d. random variables with distribution

$$\mathbb{P}(X_1 = k) = 2^{-k-2}, \qquad k \ge -1.$$

Put $\Sigma_j = \sum_{i=1}^{j} X_i$, with $\Sigma_0 = 0$. Fix $0 \le k \le \sigma$, and denote by $S^{(k)} = (S^{(k)}(j), j = 0, \dots, \sigma)$ the discrete bridge distributed as $(\Sigma_j, j = 0, \dots, \sigma)$ conditioned on $\{\Sigma_\sigma = -k\}$. Then the above considerations imply that $S^{(k)}$ is uniformly distributed over the set of all bridges in $\mathfrak{B}_\sigma$ which attain the terminal value $-k$ at time σ. Second, using that b is uniformly distributed over $\mathfrak{B}_\sigma$, we can compute

$$\mathbb{P}(\mathsf{b}(\sigma) = -k) = \frac{1}{2} \frac{(2\sigma - k - 1)!}{(2\sigma - 1)!} \frac{\sigma!}{(\sigma - k)!} \le 2^{-k}, \tag{4.8}$$

and $\mathbb{P}(\mathsf{b}(\sigma) = -k) \to 2^{-k-1}$ as $\sigma \to \infty$; see [9], Proof of Proposition 7, for a complete argument.

4.5.3. *Forests*. In the rest of this paper, we will often use the following well-known fact (see, e.g., [31], Section 2): If $\mathfrak{f} = (\tau_0, \dots, \tau_{\sigma-1})$ is chosen uniformly at random among all forests with σ trees and n edges, then the corresponding discrete contour path $(C_{\mathfrak{f}}(j), j = 0, \dots, 2n + \sigma)$, is distributed as a simple random walk path starting at 0 and conditioned to end at $-\sigma$ at time $2n + \sigma$. As a consequence, we have for $j \in \{1, \dots, \sigma\}$ and a positive integer k,

$$\mathbb{P}\left(\sum_{i=0}^{j-1} |\tau_i| = k\right) = \mathbb{P}(T_{-j} = 2k + j \mid T_{-\sigma} = 2n + \sigma), \tag{4.9}$$

where T_{-i} denotes the first hitting time of $-i$ of a simple random walk started at 0. Also note that the joint law of the trees $(\tau_0, \dots, \tau_{\sigma-1})$ is invariant under permutation of its components. Moreover, the sequence of trees $(\tau_0, \dots, \tau_{\sigma-1})$ has the law of σ independent critical geometric Galton–Watson trees conditioned to have total size n. In this context, we recall (see, e.g., [31], Section 2.2) that if $\mathbb{P}_{\mathrm{GW}}$ is the law of critical geometric Galton–Watson tree and τ a given finite tree, then

$$\mathbb{P}_{\mathrm{GW}}(\tau) = (1/2)4^{-|\tau|}. \tag{4.10}$$

Probabilities as in (4.9) can be computed using Kemperman's formula (see, e.g., [36], Chapter 6). It tells us that if $(S_i, i \in \mathbb{N}_0)$ is a simple random walk started at 0, then

$$\mathbb{P}(T_j = k) = \frac{|j|}{k}\mathbb{P}(S_k = j), \qquad j \in \mathbb{Z}, k \in \mathbb{N}. \tag{4.11}$$

By applying Kemperman's formula to $\mathbb{P}(T_{-\sigma} = 2n + \sigma)$ and counting paths, we obtain

$$\left|\mathfrak{F}_\sigma^n\right| = 3^n \frac{\sigma}{2n+\sigma}\binom{2n+\sigma}{n}.$$

Note that the factor 3^n accounts for the 3^n possible labelings of a forest with n tree edges.

For estimating $\mathbb{P}(S_k = j)$ when k and j are large, one typically applies a local central limit theorem. Setting

$$\overline{p}(k, j) = \frac{2}{\sqrt{2\pi k}} \exp\left(-\frac{j^2}{2k}\right), \qquad j \in \mathbb{Z}, k \in \mathbb{N},$$

and $\overline{p}(0, j) = \delta_0(j)$, one has (see, e.g., [28], Theorem 1.2.1)

$$\mathbb{P}(S_k = j) = \overline{p}(k, j) + O\big(1/k^{3/2}\big) \tag{4.12}$$

if $k + j$ is even, and $\mathbb{P}(S_k = j) = 0$ otherwise. Note that the above display holds uniformly in the choice of j. For us, it will mostly be sufficient to record that $\mathbb{P}(S_k = j) \le Ck^{-1/2}$ for some $C > 0$ uniformly in j and k.

However, in the boundary regime $\sigma_n \gg \sqrt{n}$, we will sometimes find ourselves in an atypical regime for simple random walk, where the control provided by (4.12) is not good enough. In this case, we use the following asymptotic expression due to Beneš [5], Theorem 1.3, first case. For $x \ll m$ such that $x + m$ is even,

$$\mathbb{P}(S_m = x) = \sqrt{\frac{2}{\pi m}} \exp\left(-\sum_{\ell=1}^{\infty} \frac{1}{2\ell(2\ell-1)} \frac{x^{2\ell}}{m^{2\ell-1}}\right)\left(1 + O\left(\frac{x^2}{m^2} + \frac{1}{m}\right)\right), \tag{4.13}$$

uniformly in (x, m). Note that as it is remarked in [5], this expression can also be obtained from [12], Theorem 6.1.6, by an explicit calculation of the rate function.

4.5.4. *Remarks on notation.* For ease of reading, let us finally specify a (notational) framework.

THE USUAL SETTING. For each $n \in \mathbb{N}$, we let $Q_n^{\sigma_n} = (V(Q_n^{\sigma_n}), d_{\mathrm{gr}}, \rho_n)$ be uniformly distributed over the set $\mathcal{Q}_n^{\sigma_n}$ of rooted quadrangulations with n internal faces and $2\sigma_n$ boundary edges. Given $Q_n^{\sigma_n}$, we choose $v_n^\bullet$ uniformly at random among the elements of $V(Q_n^{\sigma_n})$, and then $(Q_n^{\sigma_n}, v_n^\bullet)$ is uniformly distributed over $\mathcal{Q}^\bullet_{n,\sigma_n}$ and corresponds through the Bouttier–Di Francesco–Guitter bijection to a triplet $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ uniformly distributed over the set $\mathfrak{F}_{\sigma_n}^n \times \mathfrak{B}_{\sigma_n}$. We let (C_n, L_n) be the contour pair corresponding to $(\mathfrak{f}_n, \mathfrak{l}_n)$ and write

$$\mathfrak{L}_n = \big(L_n(t) + \mathsf{b}_n\big(-\underline{C}_n(t)\big), 0 \le t \le 2n + \sigma_n\big)$$

for the label function associated to $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$.

The random triplet $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ represents a uniformly labeled critical infinite forest and an independent uniform infinite bridge and encodes the UIHPQ $Q_\infty^\infty = (V(Q_\infty^\infty), d_{\mathrm{gr}}, \rho)$. We write (C_∞, L_∞) for the corresponding contour pair and $\mathfrak{L}_\infty$ for the label function.

While $B_r(Q_n^{\sigma_n})$ denotes the closed ball of radius r around the root ρ_n in $Q_n^{\sigma_n}$ (viewed as a compact metric space), we will also consider the ball $B_r^{(0)}(Q_n^{\sigma_n})$ around the vertex $\mathfrak{f}_n(0) = (0)$, and similarly for the UIHPQ.

Given a random variable (or sequence) U and an event $\mathcal{E}$, we will write $\mathcal{L}(U)$ and $\mathcal{L}(U \mid \mathcal{E})$ for the law of U and the conditional law of U given $\mathcal{E}$, respectively. The total variation norm of a probability measure is denoted by $\|\cdot\|_{\mathrm{TV}}$.

5. Auxiliary results. In this part, we collect general results and observations which will be useful later on. Our statements on Galton–Watson trees might be of some interest on its own.

5.1. *Convergence of forests.* The first two lemmas in this section provide the necessary control over the trees of a forest $\mathfrak{f}_n$ chosen uniformly at random in $\mathfrak{F}_n^{\sigma_n}$ in the regime $\sigma_n \ll \sqrt{n}$. The proofs are given in the Supplementary Material [3].

LEMMA 5.1. *Assume $\sigma_n \ll \sqrt{n}$. Denote by $(\tau_i)_{1 \le i \le \sigma_n}$ a family of σ_n independent critical geometric Galton–Watson trees. Then, for every* $0 < \delta < 1$,

$$\lim_{n\to\infty} \mathbb{P}\left(\exists i \in \{1, \dots, \sigma_n\} \text{ with } |\tau_i| \ge \delta n \,\middle|\, \sum_{i=1}^{\sigma_n} |\tau_i| = n\right) = 1.$$

LEMMA 5.2. *Assume $\sigma_n \ll \sqrt{n}$. Denote by $(\tau_i)_{1 \le i \le \sigma_n}$ a family of σ_n independent critical geometric Galton–Watson trees. Write i_* for the smallest index such that $|\tau_{i_*}| \ge \max_{1 \le i \le \sigma_n, i \ne i_*} |\tau_j|$. Then*

$$\lim_{n\to\infty} \left\| \mathcal{L}\left((\tau_i)_{1 \le i \le \sigma_n, i \ne i_*} \,\middle|\, \sum_{i=1}^{\sigma_n} |\tau_i| = n\right) - \mathcal{L}\big((\tau_i)_{1 \le i \le \sigma_n - 1}\big) \right\|_{\mathrm{TV}} = 0.$$

The next statement will prove useful for the regimes $1 \ll \sigma_n \ll \sqrt{n}$ and $\sigma_n \sim \sigma\sqrt{2n}$, $\sigma \in (0, \infty)$, as well as for the local convergence of $Q_n^{\sigma_n}$ toward the UIHPQ when $1 \ll \sigma_n \ll n$. We stress that if $\sigma_n \ll \sqrt{n}$, the following lemma is already a corollary of Lemmas 5.1 and 5.2.

LEMMA 5.3. *Assume $1 \ll \sigma_n \ll n$. Denote by $(\tau_i)_{1\le i\le\sigma_n}$ a family of σ_n independent critical geometric Galton–Watson trees. If k_n is a sequence of positive integers with $k_n \le \sigma_n$ and $k_n = o(\sigma_n \wedge (n/\sigma_n))$ as $n \to \infty$, then*

$$\lim_{n\to\infty}\left\|\mathcal{L}\left((\tau_i)_{1\le i\le k_n}\ \Big|\ \sum_{i=1}^{\sigma_n}|\tau_i| = n\right) - \mathcal{L}\big((\tau_i)_{1\le i\le k_n}\big)\right\|_{\mathrm{TV}} = 0.$$

For the proof, we refer again to the Supplementary Material [3].

5.2. *Convergence of bridges.* Here, we collect two convergence results of a bridge b_n uniformly distributed in $\mathfrak{B}_{\sigma_n}$ which are valid in all regimes $\sigma_n \gg 1$. The first lemma follows from [7], Lemma 10 (recall the remarks above on the distribution of b_n).

LEMMA 5.4. *Assume $\sigma_n \to \infty$, and let b_n be a bridge of length σ_n uniformly distributed in $\mathfrak{B}_{\sigma_n}$. Then $(\mathsf{b}_n(\sigma_n s)/\sqrt{2\sigma_n}, 0 \le s \le 1)$ converges as $n \to \infty$ to a standard Brownian bridge $\mathbb{b}$, and the convergence holds in distribution in the space $\mathcal{C}([0, 1], \mathbb{R})$.*

The next lemma provides a finer convergence without normalization for the bridge restricted to the first and last k_n values when $k_n = o(\sigma_n)$.

LEMMA 5.5. *Assume $\sigma_n \to \infty$. Let b_n be uniformly distributed in $\mathfrak{B}_{\sigma_n}$, and let b_∞ be a uniform infinite bridge as defined in Section 4.4.2. Then, if k_n is a sequence of positive integers with $k_n \le \sigma_n$ and $k_n = o(\sigma_n)$ as $n \to \infty$,*

$$\begin{aligned}\lim_{n\to\infty}\big\|&\mathcal{L}\big((\mathsf{b}_n(\sigma_n - k_n), \dots, \mathsf{b}_n(\sigma_n - 1), \mathsf{b}_n(0), \mathsf{b}_n(1), \dots, \mathsf{b}_n(k_n))\big)\\ &- \mathcal{L}\big((\mathsf{b}_\infty(-k_n), \dots, \mathsf{b}_\infty(-1), \mathsf{b}_\infty(0), \mathsf{b}_\infty(1), \dots, \mathsf{b}_\infty(k_n))\big)\big\|_{\mathrm{TV}} = 0.\end{aligned}$$

The proof of Lemma 5.5 is provided in the Supplementary Material [3].

5.3. *Root issues.* We work in the usual setting introduced in Section 4.5.4. As the next lemma shows, instead of showing distributional convergence of balls in $Q_n^{\sigma_n}$ or Q_∞^∞ around the roots, we can as well consider the corresponding balls around (0).

LEMMA 5.6. *Let $(a_n, n \in \mathbb{N})$ be a sequence of positive reals with $a_n \to \infty$ as $n \to \infty$. Let $r \ge 0$. Then, in the usual notational setting, we have the following convergences in probability as $n \to \infty$:*

(a) $d_{\mathrm{GH}}(B_r(a_n^{-1} \cdot Q_n^{\sigma_n}), B_r^{(0)}(a_n^{-1} \cdot Q_n^{\sigma_n})) \to 0$,
(b) $d_{\mathrm{GH}}(B_r(a_n^{-1} \cdot Q_\infty^\infty), B_r^{(0)}(a_n^{-1} \cdot Q_\infty^\infty)) \to 0$.

The proof will be a consequence of the following general lemma.

LEMMA 5.7. *Let $r \geq 0$, and let $\mathbf{E} = (E, d, \rho)$ and $\mathbf{E}' = (E', d', \rho')$ be two pointed complete and locally compact length spaces. Let $\mathcal{R} \subset E \times E'$ be a subset with the following properties*:

- $(\rho, \rho') \in \mathcal{R}$,
- *for all $x \in B_r(\mathbf{E})$, there exists $x' \in E'$ such that $(x, x') \in \mathcal{R}$,*
- *for all $y' \in B_r(\mathbf{E}')$, there exists $y \in E$ such that $(y, y') \in \mathcal{R}$.*

Then $d_{\mathrm{GH}}(B_r(\mathbf{E}), B_r(\mathbf{E}')) \leq (3/2)\mathrm{dis}(\mathcal{R})$.

REMARK 5.8. Note that $\mathcal{R}$ is not necessarily a correspondence; nonetheless, the definition of the distortion $\mathrm{dis}(\mathcal{R})$ from Section 2.4 makes sense (we allow it to take the value $+\infty$).

PROOF OF LEMMA 5.7. We construct a correspondence $\tilde{\mathcal{R}}$ between $B_r(\mathbf{E})$ and $B_r(\mathbf{E}')$. For each $x \in B_r(\mathbf{E})$, there exists by assumption $x' = x'_x \in E'$ such that $(x, x') \in \mathcal{R}$. Since $d'(x', \rho') \leq d(x, \rho) + \mathrm{dis}(\mathcal{R})$, we see that in fact $x' \in B_{r+\mathrm{dis}(\mathcal{R})}(\mathbf{E}')$. We choose $z' = z'(x) \in B_r(\mathbf{E}')$ that minimizes $d'(x', z')$. Note that such a z' exists in a complete and locally compact length space. Then $d'(x', z') \leq \mathrm{dis}(\mathcal{R})$. In an entirely similar way, using the third property of $\mathcal{R}$ instead of the second, we assign to each $y' \in B_r(\mathbf{E}')$ an element $z = z(y') \in B_r(\mathbf{E})$. In this notation, we now define

$$\tilde{\mathcal{R}} = \{(x, z'(x)) : x \in B_r(\mathbf{E})\} \cup \{(z(y'), y') : y' \in B_r(\mathbf{E}')\}.$$

Clearly, $\tilde{\mathcal{R}}$ is a correspondence between $B_r(\mathbf{E})$ and $B_r(\mathbf{E}')$, and a straightforward application of the triangle inequality shows that in fact $\mathrm{dis}(\tilde{\mathcal{R}}) \leq 3\mathrm{dis}(\mathcal{R})$. This proves our claim, and hence the lemma. □

PROOF OF LEMMA 5.6. We show only (a), the proof of (b) is similar. We apply Lemma 5.7 as follows. Instead of considering $(V(Q_n^{\sigma_n}), d_{\mathrm{gr}}, \rho_n)$, we may work with the corresponding pointed length space $\mathbf{E}_n = (E_n, d, \rho_n)$ obtained from replacing edges by Euclidean segments of length one, as explained in Section 2.4.2 (the distance d between two points is given by the length of a shortest path between them). Similarly, we replace $(V(Q_n^{\sigma_n}), d_{\mathrm{gr}}, (0))$ by $\mathbf{E}'_n = (E_n, d, (0))$. Define

$$\mathcal{R}_n = \{(\rho_n, (0))\} \cup \{(x, x) : x \in E_n\}.$$

Then $\mathcal{R}_n$ fulfills trivially the properties of Lemma 5.7, and we have $\mathrm{dis}(\mathcal{R}_n) \leq d(\rho_n, (0)) = -\mathsf{b}_n(\sigma_n)$ by (4.2). From (4.8) we see that $\mathsf{b}_n(\sigma_n)$ is stochastically bounded, and the claim follows. □

6. Main proofs.

6.1. *Brownian plane.* We prove Theorem 3.1, where $\sqrt{\sigma_n} \ll a_n \ll n^{1/4}$.

IDEA OF THE PROOF. Let $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ be uniformly distributed over $\mathfrak{F}^n_{\sigma_n} \times \mathfrak{B}_{\sigma_n}$. Thanks to Lemmas 5.1 and 5.2, we know that for large n, $\mathfrak{f}_n$ has a unique largest tree of order n, and all the other $\sigma_n - 1$ trees behave as independent critical geometric Galton–Watson trees. The maximal label in these $\sigma_n - 1$ nonlargest trees is of order $\sqrt{\sigma_n}$; see Lemma 6.2. Upon rescaling distances by a_n^{-1}, this implies by a result of Bettinelli [9], Lemma 23, that the part of the quadrangulation encoded by the forest without its largest tree τ is negligible in the limit $n \to \infty$. Conditionally on its size, τ is uniformly distributed among all plane trees, and so is the associated quadrangulation among all quadrangulations with $|\tau|$ faces and no boundary. Now the second part of [20], Theorem 2, applies, stating that the Brownian plane is the scaling limit $m \to \infty$ of uniform quadrangulations with m faces when the scaling grows slower than $m^{1/4}$.

To make things precise, we recall the following.

LEMMA 6.1 (Lemma 23 in [9]). *Let $\sigma \in \mathbb{N}$. Let $((\mathfrak{f}, \mathfrak{l}), \mathsf{b}) \in \mathfrak{F}^n_\sigma \times \mathcal{B}_\sigma$. Fix any tree τ of $\mathfrak{f}$. Let $b \in \{-1, 0\}$. We view $(\tau, \mathfrak{l} \restriction \tau)$ as an element of $\mathfrak{F}^{|\tau|}_1$ and denote by $\mathfrak{q}_\mathfrak{f} \in \mathcal{Q}^\sigma_n$ and $\mathfrak{q}_\tau \in \mathcal{Q}^1_{|\tau|}$ the quadrangulations associated to $((\mathfrak{f}, \mathfrak{l}), \mathsf{b})$ and $((\tau, \mathfrak{l} \restriction \tau), (0, b))$, respectively, through the Bouttier–Di Francesco–Guitter bijection (the distinguished vertices are omitted). Then*

$$d_{\mathrm{GH}}\big(V(\mathfrak{q}_\mathfrak{f}), V(\mathfrak{q}_\tau)\big) \le 2\Big(\max_{V(\mathfrak{f}\setminus\mathring{\tau})} \hat{\mathfrak{l}} - \min_{V(\mathfrak{f}\setminus\mathring{\tau})} \hat{\mathfrak{l}} + 1\Big),$$

where $\mathring{\tau}$ stands for the tree τ without its root vertex, and

$$\hat{\mathfrak{l}}(u) = \mathfrak{l}(u) + \mathsf{b}\big(\mathfrak{r}(u)\big), \qquad u \in V(\mathfrak{f}),$$

is the labeling of $\mathfrak{f}$ shifted by the values of the bridge b (see Section 4.1).

Let $r \ge 0$. For the balls $B_r(\mathfrak{q}_\mathfrak{f})$ and $B_r(\mathfrak{q}_\tau)$ around the root vertices, we claim that

$$d_{\mathrm{GH}}\big(B_r(\mathfrak{q}_\mathfrak{f}), B_r(\mathfrak{q}_\tau)\big) \le 3 d_{\mathrm{GH}}\big(V(\mathfrak{q}_\mathfrak{f}), V(\mathfrak{q}_\tau)\big) + 8. \tag{6.1}$$

Indeed, we may first replace both $V(\mathfrak{q}_\mathfrak{f})$ and $V(\mathfrak{q}_\tau)$ by the corresponding length spaces $\mathbf{Q}_\mathfrak{f}$ and $\mathbf{Q}_\tau$ as explained in Section 2.4.2. We obtain

$$\big|d_{\mathrm{GH}}\big(B_r(\mathfrak{q}_\mathfrak{f}), B_r(\mathfrak{q}_\tau)\big) - d_{\mathrm{GH}}\big(B_r(\mathbf{Q}_\mathfrak{f}), B_r(\mathbf{Q}_\tau)\big)\big| \le 2.$$

We now note that every correspondence between $\mathbf{Q}_{\mathfrak{f}}$ and $\mathbf{Q}_{\tau}$ satisfies the requirements of Lemma 5.7, so that by this lemma

$$d_{\mathrm{GH}}\big(B_r(\mathbf{Q}_{\mathfrak{f}}), B_r(\mathbf{Q}_{\tau})\big) \le (3/2) \inf_{\mathcal{R}} \operatorname{dis}(\mathcal{R}) = 3 d_{\mathrm{GH}}(\mathbf{Q}_{\mathfrak{f}}, \mathbf{Q}_{\tau}) \\ \le 3 d_{\mathrm{GH}}\big(V(\mathfrak{q}_{\mathfrak{f}}), V(\mathfrak{q}_{\tau})\big) + 6,$$

where the infimum is taken over all correspondences between $\mathbf{Q}_{\mathfrak{f}}$ and $\mathbf{Q}_{\tau}$. The preceding arguments give (6.1). We are now in position to prove Theorem 3.1.

PROOF OF THEOREM 3.1. We have to show that for each $r \ge 0$,

$$B_r\big(a_n^{-1} \cdot Q_n^{\sigma_n}\big) \xrightarrow[n\to\infty]{(d)} B_r(\mathsf{BP}) \tag{6.2}$$

in distribution in $\mathbb{K}$. Let $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ be uniformly distributed in $\mathfrak{F}_{\sigma_n}^n \times \mathfrak{B}_{\sigma_n}$, and denote by $\tau_*^{(n)}$ the largest tree of $\mathfrak{f}_n$ (we take that with the smallest index if several trees attain the largest size). We let $b_n \in \{-1, 0\}$ be uniformly distributed and independent of everything else and denote by $\hat{Q}_n$ the quadrangulation encoded by $((\tau_*^{(n)}, \mathfrak{l}_n \restriction \tau_*^{(n)}), (0, b_n))$, in the same way as in Lemma 6.1. Assuming as usual that $Q_n^{\sigma_n}$ is encoded by $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$, we obtain from (6.1) together with Lemma 6.1 that

$$d_{\mathrm{GH}}\big(B_r\big(a_n^{-1} \cdot Q_n^{\sigma_n}\big), B_r\big(a_n^{-1} \cdot \hat{Q}_n\big)\big) \le \frac{6}{a_n}\Big(\max_{V(\mathfrak{f}_n \setminus \mathring{\tau}_*^{(n)})} \hat{\mathfrak{l}}_n - \min_{V(\mathfrak{f}_n \setminus \mathring{\tau}_*^{(n)})} \hat{\mathfrak{l}}_n\Big) + o(1) \tag{6.3}$$

as $n \to \infty$, where in the notation of Lemma 6.1, $\mathring{\tau}_*^{(n)}$ stands for the tree $\tau_*^{(n)}$ without its root, and $\hat{\mathfrak{l}}_n(u) = \mathfrak{l}_n(u) + \mathsf{b}_n(\mathfrak{r}(u))$, $u \in V(\mathfrak{f}_n)$, is the labeling of $\mathfrak{f}_n$ shifted by b_n. We claim that the right-hand side of (6.3) converges to zero in probability. In this regard, recall that by Lemma 5.4, the values of b_n are of order $\sqrt{\sigma_n} \ll a_n$, so that we may replace $\hat{\mathfrak{l}}_n$ by $\mathfrak{l}_n$ in (6.3). Denote by $\mathfrak{f}_n' = \mathfrak{f}_n \setminus \tau_*^{(n)}$ the forest obtained from $\mathfrak{f}_n$ by removing $\tau_*^{(n)}$, that is, if $\tau_*^{(n)}$ is the tree of $\mathfrak{f}_n$ with index i, then $\mathfrak{f}_n' = (\tau_0^{(n)}, \dots, \tau_{i-1}^{(n)}, \tau_{i+1}^{(n)}, \dots, \tau_{\sigma_n - 1}^{(n)})$. We write (C_n', L_n') for the contour pair corresponding to $(\mathfrak{f}_n', \mathfrak{l}_n \restriction \mathfrak{f}_n')$. We view both C_n' and L_n' as continuous functions on $[0, \infty)$ by letting $C_n'(s) = C_n'(s \wedge (2(n - |\tau_*^{(n)}|) + \sigma_n - 1))$, and similarly with L_n'. The convergence to zero of the right-hand side in (6.3) follows now from the following lemma.

LEMMA 6.2. *In the notation from above, if $a_n \gg \sqrt{\sigma_n}$, then for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}\Big(\sup_{t\ge 0} \frac{1}{a_n} |L_n'(t)| \ge \varepsilon\Big) = 0.$$

PROOF. Let $(\tilde{\tau}_i, (\tilde{\ell}_i(u))_{u \in V(\tau_i)})$, $i = 0, \dots, \sigma_n - 2$, be a sequence of $\sigma_n - 1$ uniformly labeled critical geometric Galton–Watson trees. Consider the forest $\tilde{\mathfrak{f}}_n =$

$(\tilde{\tau}_0, \dots, \tilde{\tau}_{\sigma_n - 2})$ together with the labeling $\tilde{\mathfrak{l}}_n$ given by $\tilde{\mathfrak{l}}_n \restriction V(\tilde{\tau}_i) = \tilde{\ell}_i$, for all i. Let $(\tilde{C}_n, \tilde{L}_n)$ denote the contour pair associated to $(\tilde{\mathfrak{f}}_n, \tilde{\mathfrak{l}}_n)$, continuously extended to $[0, \infty)$ outside $[0, 2\sum_{i=0}^{\sigma_n - 2} |\tilde{\tau}_i| + \sigma_n - 1]$ as described above.

By Lemma 5.2, we can for each $\delta > 0$ couple the pairs (C_n', L_n') and $(\tilde{C}_n, \tilde{L}_n)$ on the same probability space such that with probability at least $1 - \delta$, we have the equality $(C_n', L_n') = (\tilde{C}_n, \tilde{L}_n)$ as elements of $\mathcal{C}([0, \infty), \mathbb{R})^2$, provided n is sufficiently large. Our claim therefore follows if

$$\lim_{n \to \infty} \mathbb{P}\left(\sup_{t \geq 0} \frac{1}{a_n} |\tilde{L}_n(t)| \geq \varepsilon \right) = 0. \tag{6.4}$$

From Section 4.5.3, we know that $\tilde{C}_n$ has the law of a simple random walk started from 0 and stopped upon hitting $-(\sigma_n - 1)$, with linear interpolation between integer values. By Donsker's invariance principle, we know that $((1/\sigma_n)\tilde{C}_n(\sigma_n^2 t), t \geq 0)$ converges in distribution to a Brownian motion $(B_{t \wedge T_{-1}}, t \geq 0)$ stopped upon hitting -1. Arguments like in [31], proof of Theorem 4.3, then imply convergence of the finite-dimensional laws on $\mathcal{C}([0, \infty), \mathbb{R}^2)$ of the process $((1/\sigma_n)\tilde{C}_n(\sigma_n^2 \cdot), (1/\sqrt{\sigma_n})\tilde{L}_n(\sigma_n^2 \cdot))$, and tightness of the second component follows *via* Kolmogorov's criterion from moment bounds on $\tilde{C}_n$ as in [31], Lemma 2.13, (in our case, these bounds are in fact easier to establish, since we consider an unconditioned random walk). We do not repeat the arguments here, but refer the reader to [31] or [7], Section 5, for more details. We obtain the convergence in distribution

$$\left(\frac{1}{\sigma_n} \tilde{C}_n(\sigma_n^2 \cdot), \frac{1}{\sqrt{\sigma_n}} \tilde{L}_n(\sigma_n^2 \cdot) \right) \xrightarrow[n \to \infty]{(d)} (B_{\cdot \wedge T_{-1}}, Z) \qquad \text{in } \mathcal{C}([0, \infty), \mathbb{R}^2),$$

where $Z = (Z_t, t \geq 0)$ is the Brownian snake driven by $(B_{t \wedge T_{-1}}, t \geq 0)$. Since $a_n \gg \sqrt{\sigma_n}$, this last result clearly implies (6.4) and hence the assertion of the lemma. $\square$

Going back to (6.3), it remains to show that for $\varepsilon > 0$, $F : \mathbb{K} \to \mathbb{R}$ continuous and bounded and $n \geq n_0$,

$$\left|\mathbb{E}\left[F\left(B_r(a_n^{-1} \cdot \hat{Q}_n)\right)\right] - \mathbb{E}\left[F\left(B_r(\mathsf{BP})\right)\right]\right| \leq \varepsilon. \tag{6.5}$$

Let $0 < \delta < 1$. We estimate

$$\begin{aligned}
&\left|\mathbb{E}\left[F\left(B_r(a_n^{-1} \cdot \hat{Q}_n)\right)\right] - \mathbb{E}\left[F\left(B_r(\mathsf{BP})\right)\right]\right| \\
&\quad \leq 2 \sup |F| \mathbb{P}\left(\left|\tau_*^{(n)}\right| \leq \delta n\right) \\
&\qquad + \sum_{k=\lceil \delta n \rceil}^{n} \mathbb{P}\left(\left|\tau_*^{(n)}\right| = k\right) \left|\mathbb{E}\left[F\left(B_r(a_n^{-1} \cdot \hat{Q}_n)\right) \mid \left|\tau_*^{(n)}\right| = k\right]\right. \\
&\qquad \left. - \mathbb{E}\left[F\left(B_r(\mathsf{BP})\right)\right]\right|.
\end{aligned}$$

For $n \geq n(\delta, \varepsilon) \in \mathbb{N}$ sufficiently large, Lemma 5.1 gives $2 \sup |F| \mathbb{P}(|\tau_*^{(n)}| \leq \delta n) \leq \varepsilon/2$. As to the sum, we note that conditionally on $|\tau_*^{(n)}| = k$, $\hat{Q}_n$ is uniformly distributed among all rooted quadrangulations with k inner faces and a boundary of size 2. Removing the only edge of the boundary which is not the root edge, we obtain a uniform quadrangulation with k faces and no boundary. Clearly, the removal of this edge does not change the underlying metric space. By [20], Theorem 2, we therefore get for $k \geq \lceil \delta n \rceil$ and n sufficiently large, recalling that $a_n \ll n^{1/4}$,

$$\left|\mathbb{E}\left[F\left(B_r\left(a_n^{-1} \cdot \hat{Q}_n\right)\right)\right] \mid |\tau_*^{(n)}| = k\right] - \mathbb{E}\left[F\left(B_r(\mathsf{BP})\right)\right]\right| \leq \varepsilon/2.$$

This shows (6.5), and hence (6.2). $\square$

6.2. *Coupling of Brownian disk and half-planes*. We will now prove Theorem 3.7 and Corollary 3.8. Let us first show how the corollary follows from the theorem.

PROOF OF COROLLARY 3.8. Theorem 3.7 implies that with probability 1, for every $r \geq 0$, the ball $B_r(\mathsf{BHP}_\theta)$ is included in an open set of BHP_θ homeomorphic to $\overline{\mathbb{H}}$. This shows that BHP_θ is a simply connected topological surface with a boundary, and that this boundary is connected and noncompact: it must therefore be homeomorphic to $\mathbb{R}$. We construct a surface S without boundary by gluing a copy H of the closed half-plane $\overline{\mathbb{H}}$ to BHP_θ along the boundary. This noncompact surface is still simply connected by van Kampens' theorem (see Theorem 1.20 in [25]), and in particular, it is one-ended. Therefore, it must be homeomorphic to $\mathbb{R}^2$; see [37]. Now if ϕ is a homeomorphism from the boundary of BHP_θ to $\mathbb{R}$, then a simple variation of the Jordan–Schoenflies theorem (see, e.g., [40], Theorem 3.1) implies that ϕ can be extended to a homeomorphism $\overline{\phi}$ from S to $\mathbb{R}^2$, and the two halves BHP_θ and H of S must be sent *via* $\overline{\phi}$ to the two half-spaces $\overline{\mathbb{H}}$ and $-\overline{\mathbb{H}}$. In particular, $\overline{\phi}$ induces a homeomorphism from BHP_θ to a closed half-plane, as wanted. $\square$

We turn to Theorem 3.7, and in this regard, we first collect some notation used throughout this section.

6.2.1. *Notation*: *Brownian half-plane and disk*. We fix a perimeter function $\sigma(\cdot) : (0, \infty) \to (0, \infty)$ as given in the statement of Theorem 3.7 and let $\theta = \lim_{T \to \infty} \sigma(T)/T \in [0, \infty)$. For $T > 0$ large but fixed, we will work with the following processes, which we tacitly assume to be defined on a joint probability space:

- F a first passage Brownian bridge on $[0, T]$ from 0 to $-\sigma(T)$;
- b a Brownian bridge on $[0, \sigma(T)]$ from 0 to 0, scaled by $\sqrt{3}$, independent of F;
- B a Brownian motion on $[0, \infty)$ with drift $-\theta$, started from $B_0 = 0$;
- Π the Pitman transform of an independent copy of B;

- γ a two-sided Brownian motion on $\mathbb{R}$ with $\gamma_0 = 0$, scaled by the factor $\sqrt{3}$, independent of (B, Π).

Recall Definition 2.3. We will assume that the *Brownian disk* $\mathsf{BD}_{T,\sigma(T)}$ is given in terms of the contour function F, and that its label function W is defined by

$$(W_t, 0 \le t \le T) = (b_{-\underline{F}_t} + Z_t, 0 \le t \le T),$$

where $(Z_t, 0 \le t \le T) = Z^{F-\underline{F}}$ is the random snake driven by $F - \underline{F}$, with $\underline{F}_t = \inf_{[0,t]} F$. We shall write d_F and d_W for the pseudo-metrics on $I = [0, T]$ associated to F and W, respectively; cf. (2.1). Moreover, we write D instead of $D_{F,W}$ (cf. (2.2)), so that the Brownian disk $\mathsf{BD}_{T,\sigma(T)}$ is the pointed metric space $([0, T]/\{D = 0\}, D, \rho)$, with ρ being the equivalence class of 0.

REMARK 6.3. We stress that all the quantities appearing in the definition of $\mathsf{BD}_{T,\sigma(T)}$ depend on T or $\sigma(T)$ (like F, b, W, Z or the pseudo-metric D). The real T measuring the volume will be chosen sufficiently large later on, but for the ease of reading, we mostly suppress T from the notation. Note moreover that we *define* here W in terms of the processes F, b and Z; in particular, W does not denote the canonical process as in Section 2. Finally, we stress that the factor $\sqrt{3}$ is already part of the definition of γ, contrary to Definition 2.3.

Next, recall Definition 2.6. We will assume that the *Brownian half-plane* BHP_θ, $\theta \in [0, \infty)$, is given in terms of contour and label processes $X^\theta = (X^\theta_t, t \in \mathbb{R})$ and $W^\theta = (W^\theta_t, t \in \mathbb{R})$ defined as follows:

$$X^\theta_t = \begin{cases} B_t & \text{if } t \ge 0, \\ \Pi_t & \text{if } t < 0, \end{cases} \qquad W^\theta = \left(\gamma_{-\underline{X}^\theta_t} + Z^\theta_t, t \in \mathbb{R}\right),$$

where $Z^\theta = (Z^\theta_t, t \in \mathbb{R}) = Z^{X^\theta - \underline{X}^\theta}$ is the random snake driven by $X^\theta - \underline{X}^\theta$, with $\underline{X}^\theta_t = \inf_{[0,t]} X^\theta$ for $t \ge 0$, and $\underline{X}^\theta_t = \inf_{(-\infty,t]} X^\theta$ for $t < 0$. We write d_{X^θ} and d_{W^θ} for the pseudo-metrics on $\mathbb{R}$ associated to X^θ and W^θ, respectively, and D_θ instead of D_{X^θ,W^θ}, so that the Brownian half-plane BHP_θ is given by the pointed metric space $(\mathbb{R}/\{D_\theta = 0\}, D_\theta, \rho_\theta)$, with ρ_θ being the equivalence class of 0.

6.2.2. *Absolute continuity relation between contour functions*. A key step in proving Theorem 3.7 is to relate the contour function X^θ for BHP_θ to the contour function F for $\mathsf{BD}_{T,\sigma(T)}$, in spirit of [20], Proposition 3.

Let $T > 0$, and $\alpha, \beta > 0$ with $\alpha + \beta < T$. We interpret the pair $((F_t)_{0 \le t \le \alpha}, (F_{T-t})_{0 \le t \le \beta})$ as an element of the space $\mathcal{C}([0, \alpha], \mathbb{R}) \times \mathcal{C}([0, \beta], \mathbb{R})$ and write (ω, ω') for a generic element of this space. We next introduce some probability kernels. Let $t > 0$. For $x \in \mathbb{R}$, the heat kernel is denoted

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

For $x, y > 0$, the transition density of Brownian motion killed upon hitting 0 is given by

$$p_t^*(x, y) = p_t^*(y, x) = p_t(y - x) - p_t(y + x).$$

The density of the first hitting time of level $x > 0$ of Brownian motion started at 0 is

$$g_t(x) = \frac{x}{t} p_t(x).$$

The transition density of a three-dimensional Bessel process takes the form

$$r_t(x, y) = \begin{cases} 2y g_t(y) & \text{if } x = 0, \\ x^{-1} p_t^*(x, y) y & \text{if } x, y > 0. \end{cases} \tag{6.6}$$

In [38], Theorem 1, Pitman and Rogers show that the Pitman transform of a one-dimensional Brownian motion with drift $-\theta$ has the law of the radial part of a three-dimensional Brownian motion with a drift of magnitude θ. In particular, if $\theta = 0$, it has the law of a three-dimensional Bessel process, and for all $\theta \geq 0$, it is a transient process. In [38], Theorem 3, it is moreover shown that its transition density is given by

$$q_t^{(\theta)}(x, y) = \exp\bigl(-(t/2)\theta^2\bigr) h^{-1}(x\theta) r_t(x, y) h(y\theta), \tag{6.7}$$

where

$$h(x) = \begin{cases} x^{-1} \sinh x & \text{if } x > 0, \\ 1 & \text{if } x = 0. \end{cases}$$

LEMMA 6.4. *In the notation from above* (*and from Section* 6.2.1), *the law of*

$$\bigl((F_t)_{0 \leq t \leq \alpha}, (F_{T-t})_{0 \leq t \leq \beta}\bigr)$$

is absolutely continuous with respect to the law of

$$\bigl((B_t)_{0 \leq t \leq \alpha}, \bigl(\Pi_t - \sigma(T)\bigr)_{0 \leq t \leq \beta}\bigr),$$

with density given by the function

$$\begin{aligned} &\varphi_{T,\alpha,\beta}\bigl(\omega, \omega'\bigr) \\ &\quad = \mathbb{1}_{\{\omega_s > -\sigma(T) \text{ for } s \in [0,\alpha]\}}(\omega) \\ &\qquad \times \frac{p^*_{T-(\alpha+\beta)}(\omega_\alpha + \sigma(T), \omega'_\beta + \sigma(T))}{2(\omega'_\beta + \sigma(T)) g_T(\sigma(T))} \frac{\exp(\omega_\alpha \theta + \frac{\alpha+\beta}{2}\theta^2)}{h((\omega'_\beta + \sigma(T))\theta)}. \end{aligned}$$

Moreover, if $\mathbb{P}_{\alpha,\beta}$ *denotes the joint* (*product*) *law of* $((B_t)_{0 \leq t \leq \alpha}, (\Pi_t)_{0 \leq t \leq \beta})$, *the following holds true*: *For each* $\varepsilon > 0$, *there exists* $T_0 > 0$ *and a measurable set* $E = E(\varepsilon, T_0) \subset \mathcal{C}([0, \alpha], \mathbb{R}) \times \mathcal{C}([0, \beta], \mathbb{R})$ *with* $\mathbb{P}_{\alpha,\beta}(E) \geq 1 - \varepsilon$ *such that for* $T \geq T_0$,

$$\sup_{(\omega, \omega' + \sigma(T)) \in E} \bigl|\varphi_{T,\alpha,\beta}\bigl(\omega, \omega'\bigr) - 1\bigr| \leq \varepsilon.$$

Note that $\varphi_{T,\alpha,\beta}$ depends on the second coordinate ω' only through its endpoint ω'_β.

PROOF OF LEMMA 6.4. First, note that the law of the first passage Brownian bridge F is specified by $F_T = -\sigma(T)$ and

$$\mathbb{E}\big[f\big((F_t)_{0\le t\le T'}\big)\big] = \mathbb{E}\bigg[f\big((\hat\gamma)_{0\le t\le T'}\big)\mathbb{1}_{\{\underline{\hat\gamma}_{T'}>-\sigma(T)\}}\frac{g_{T-T'}(\hat\gamma_{T'}+\sigma(T))}{g_T(\sigma(T))}\bigg] \tag{6.8}$$

for all $0 \le T' < T$ and all functions $f \in \mathcal{C}([0,T'],\mathbb{R})$, where $\hat\gamma$ is a one-dimensional Brownian motion started from zero (without drift). Let us next simplify notation. For $x \in \mathbb{R}$, write $\tilde x = x + \sigma(T)$. For $0 < t_1 < t_2 < \cdots < t_p$ and $x_1, \dots, x_p > -\sigma(T)$, let

$$G_{t_1,\dots,t_p}(x_1,\dots,x_p) = p^*_{t_1}\big(\sigma(T),\tilde x_1\big)p^*_{t_2-t_1}(\tilde x_1,\tilde x_2)\cdots p^*_{t_p-t_{p-1}}(\tilde x_{p-1},\tilde x_p).$$

For $0 < t'_1 < t'_2 < \cdots < t'_q$ and $x_{p+1}, \dots, x_{p+q} > -\sigma(T)$, let

$$\begin{aligned} &H_{t'_1,\dots,t'_q}(x_{p+q},\dots,x_{p+1}) \\ &\quad = g_{t'_1}(\tilde x_{p+q})p^*_{t'_2-t'_1}(\tilde x_{p+q},\tilde x_{p+q-1})\cdots p^*_{t'_q-t'_{q-1}}(\tilde x_{p+2},\tilde x_{p+1}). \end{aligned}$$

Now fix $0 < t_1 < t_2 < \cdots < t_p = \alpha$ and $0 < t'_1 < t'_2 < \cdots < t'_q = \beta$. We infer from (6.8) that the density of the $(p+q)$-tuple $(F_{t_1}, \dots, F_{t_p}, F_{T-t'_q}, \dots, F_{T-t'_1})$ is given by the function

$$\begin{aligned} &f_{t_1,\dots,t_p,t'_1,\dots,t'_q}(x_1,\dots,x_{p+q}) \\ &\qquad = G_{t_1,\dots,t_p}(x_1,\dots,x_p)H_{t'_1,\dots,t'_q}(x_{p+q},\dots,x_{p+1}) \\ &\qquad\quad \cdot\frac{p^*_{T-(\alpha+\beta)}(\tilde x_p,\tilde x_{p+1})}{g_T(\sigma(T))}. \end{aligned} \tag{6.9}$$

From Girsanov's theorem, we know that the finite-dimensional distributions $(B_{t_1}, \dots, B_{t_p})$ of a one-dimensional Brownian motion B with drift $-\theta$ are absolutely continuous with respect to those of Brownian motion $\hat\gamma$ without drift, with a density given by $\exp(-\theta\hat\gamma_{t_p} - \alpha\theta^2/2)$. Next, we see from (6.7) that the law of $(\Pi_{t'_1} - \sigma(T), \dots, \Pi_{t'_q} - \sigma(T))$ has density

$$\begin{aligned} &\pi_{t'_1,\dots,t'_q}(x_{p+q},\dots,x_{p+1}) \\ &\quad = 2\tilde x_{p+1}\exp\big(-(\beta/2)\theta^2\big)h(\tilde x_{p+1}\theta)H_{t'_1,\dots,t'_q}(x_{p+q},\dots,x_{p+1}), \end{aligned}$$

for $x_{p+q}, \dots, x_{p+1} > -\sigma(T)$. By (6.9) and the last two observations, the first claim of the statement follows.

As for the second claim, for every $\delta > 0$, by continuity of B and Π, we can find a constant $K = K(\delta,\alpha,\beta) > 0$ in such a way that

$$\mathbb{P}\Big(\min_{[0,\alpha]} B > -K, \max_{[0,\beta]} \Pi < K\Big) \ge 1-\delta. \tag{6.10}$$

The second claim now follows from (6.10) and the fact that for every $\delta' > 0$, if T is large enough, we have

$$\left|\frac{p^*_{T-(\alpha+\beta)}(x+\sigma(T), y+\sigma(T))}{2(y+\sigma(T))g_T(\sigma(T))}\frac{\exp(x\theta+\frac{\alpha+\beta}{2}\theta^2)}{h((y+\sigma(T))\theta)}-1\right| \le \delta' \tag{6.11}$$

uniformly in $x \in \mathbb{R}$ with $|x| \le K$ and $y \ge -\sigma(T)$ with $|y+\sigma(T)| \le K$. The last display in turn follows from a straightforward but somewhat tedious calculation; we give some indication for the case $\lim_{T\to\infty} \sigma(T)/T = \theta > 0$. First, as $T \to \infty$,

$$\begin{aligned}
&\frac{p^*_{T-(\alpha+\beta)}(x+\sigma(T), y+\sigma(T))}{2(y+\sigma(T))g_T(\sigma(T))}\frac{\exp(x\theta+\frac{\alpha+\beta}{2}\theta^2)}{h((y+\sigma(T))\theta)}\\
&\quad\sim \left(\frac{\exp(-\frac{(y-x)^2}{2(T-(\alpha+\beta))})-\exp(-\frac{(x+y+2\sigma(T))^2}{2(T-(\alpha+\beta))})}{\exp(-\sigma^2(T)/(2T))}\right)\frac{\exp(x\theta+\frac{\alpha+\beta}{2}\theta^2)}{2\sinh(\theta(y+\sigma(T)))}.
\end{aligned}$$

Then, uniformly in x and y as specified above, we find

$$\exp\left(-\frac{(y-x)^2}{2(T-(\alpha+\beta))}+\frac{\sigma^2(T)}{2T}\right) \sim \exp\left((-x+y+\sigma(T))\theta-\frac{\alpha+\beta}{2}\theta^2\right),$$

and

$$\exp\left(-\frac{(x+y+2\sigma(T))^2}{2(T-(\alpha+\beta))}+\frac{\sigma^2(T)}{2T}\right) \sim \exp\left((-x-y-\sigma(T))\theta-\frac{\alpha+\beta}{2}\theta^2\right).$$

Putting these three estimates together, (6.11) follows. The remaining case $\lim_{T\to\infty}\sigma(T)/T = 0$ with $\liminf_{T\to\infty}\sigma(T)/\sqrt{T} > 0$ is similar but easier (note that the expression for $\varphi_{T,\alpha,\beta}$ simplifies when $\theta = 0$). $\square$

We need a similar absolute continuity property for the Brownian bridge b. In the next lemma, we let additionally γ' be an independent copy of the linear Brownian motion γ (again scaled by the factor $\sqrt{3}$).

LEMMA 6.5. *The law of* $((b_t)_{0\le t\le\alpha}, (b_{\sigma(T)-t})_{0\le t\le\beta})$ *is absolutely continuous with respect to the law of* $((\gamma_t)_{0\le t\le\alpha}, (\gamma'_t)_{0\le t\le\beta})$, *with density given by the function*

$$\tilde{\varphi}_{T,\alpha,\beta}(\omega,\omega') = \frac{p_{\sigma(T)-(\alpha+\beta)}((\omega'_\beta-\omega_\alpha)/\sqrt{3})}{p_{\sigma(T)}(0)}.$$

Moreover, if $\mathbb{P}_{\alpha,\beta}$ *denotes the joint law of* $((\gamma_t)_{0\le t\le\alpha}, (\gamma'_t)_{0\le t\le\beta})$, *the following holds true: For each* $\varepsilon > 0$, *there is* $T_0 > 0$ *and a measurable set* $E = E(\varepsilon, T_0) \subset \mathcal{C}([0,\alpha],\mathbb{R}) \times \mathcal{C}([0,\beta],\mathbb{R})$ *with* $\mathbb{P}_{\alpha,\beta}(E) \ge 1-\varepsilon$ *such that for* $T \ge T_0$,

$$\sup_{(\omega,\omega')\in E} \left|\tilde{\varphi}_{T,\alpha,\beta}(\omega,\omega')-1\right| \le \varepsilon.$$

PROOF. The first part is immediate from the fact that the law of the Brownian bridge b is specified by $b_{\sigma(T)} = 0$ and

$$\mathbb{E}\big[f\big((b_t)_{0\le t\le T'}\big)\big] = \mathbb{E}\bigg[f\big((\gamma_t)_{0\le t\le T'}\big)\frac{p_{\sigma(T)-T'}(\gamma_{T'}/\sqrt{3})}{p_{\sigma(T)}(0)}\bigg] \tag{6.12}$$

for all $0 \le T' < \sigma(T)$ and all $f \in \mathcal{C}([0,T'],\mathbb{R})$. The proof of the second part is very similar to that of Lemma 6.4. We omit the details. □

6.2.3. *Cactus bounds for* $\mathsf{BD}_{T,\sigma(T)}$ *and* BHP_θ. We shall need the continuous analog of the cactus bound (4.3) for the pseudo-metric D associated to the Brownian disk $\mathsf{BD}_{T,\sigma(T)}$. Recall from Section 2.1 that the contour function F encodes a random real tree $(\mathcal{T}_F, d_F)$. We write $p_F : [0,T] \to \mathcal{T}_F$ for the canonical projection. Being almost surely a class function for the equivalence relation $\{d_F = 0\}$, we may view the label function W as well as a (random) function on $\mathcal{T}_F$. In analogy to (4.3), given $0 \le s \le t \le T$, we denote by $[\![s,t]\!]_{\mathcal{T}_F}$ the geodesic segment between $p_F(s)$ and $p_F(t)$ in the tree $\mathcal{T}_F$, whereas $[\![t,s]\!]_{\mathcal{T}_F}$ stands for the union of the geodesic segments from $p_F(t)$ to $p_F(T)$ and from $p_F(0)$ to $p_F(s)$. The cactus bounds now reads

$$D(s,t) \ge W_s + W_t - 2\max\Big\{\min_{[\![s,t]\!]_{\mathcal{T}_F}} W, \min_{[\![t,s]\!]_{\mathcal{T}_F}} W\Big\}, \qquad s,t \in [0,T]. \tag{6.13}$$

See, for example, [20] for a proof of the corresponding bound in the context of the Brownian map, which can easily be adapted to the Brownian disk. We will often use the fact that for $s \le t$, $[\![s,t]\!]_{\mathcal{T}_F}$ contains all the vertices of the form $p_F(r_*)$, where r_* is a time between s and t where F attains a new minimum. In fact, $\mathcal{T}_F$ has a distinguished geodesic segment of length $\sigma(T)$ given by $p_F(\{T_y, 0 \le y \le \sigma(T)\})$, where we have set here $T_y = \inf\{r \ge 0 : F_r = -y\}$. One may view this segment as the floor of a forest of $\mathbb{R}$-trees of the form $\mathcal{T}_y = p_F((T_{y-}, T_y])$ coded by the excursions of F above its past infimum. One may then imagine the $\mathbb{R}$-tree $\mathcal{T}_y$ as being attached to the point $p_F(T_y)$ of the floor. Note that $p_F(T_y)$ is at distance y from $p_F(0)$ (in $\mathcal{T}_F$). See also [11], Section 2.

A similar bound holds for the pseudo-metric D_θ associated to the Brownian half-plane BHP_θ, namely

$$D_\theta(s',t') \ge W^\theta_{s'} + W^\theta_{t'} - 2\min_{[\![s',t']\!]_{\mathcal{T}_{X^\theta}}} W^\theta, \qquad s' \le t' \in \mathbb{R}. \tag{6.14}$$

Here, in hopefully obvious notation, $[\![s',t']\!]_{\mathcal{T}_{X^\theta}}$ stand for the geodesic segment between $p_{X^\theta}(s')$ and $p_{X^\theta}(t')$ in the (infinite) random tree $\mathcal{T}_{X^\theta}$, and $p_{X^\theta} : \mathbb{R} \to \mathcal{T}_{X^\theta}$ is the canonical projection. We refer again to the proof in [20], which can be transferred to our setting.

6.2.4. *Isometry of balls in* $\mathsf{BD}_{T,\sigma(T)}$ *and* BHP_θ. Before finally proving Theorem 3.7, we first prove the following weaker statement (compare with Proposition 4 of [20] for the Brownian map and the Brownian plane).

PROPOSITION 6.6. *Let* $\varepsilon > 0$, $r \geq 0$. *Let* $\sigma(\cdot) : (0,\infty) \to (0,\infty)$ *be a function satisfying* $\lim_{T\to\infty} \sigma(T)/T = \theta \in [0,\infty)$ *and, in case* $\theta = 0$, $\liminf_{T\to\infty} \sigma(T)/\sqrt{T} > 0$. *Then there exists* $T_0 = T_0(\varepsilon, r, \sigma)$ *such that for all* $T \geq T_0$, *one can construct copies of* $\mathsf{BD}_{T,\sigma(T)}$ *and* BHP_θ *on the same probability space such that with probability at least* $1-\varepsilon$, *the balls* $B_r(\mathsf{BD}_{T,\sigma(T)})$ *and* $B_r(\mathsf{BHP}_\theta)$ *of radius r around the respective roots are isometric.*

PROOF. We use the notation specified in Section 6.2.1. For $x \in \mathbb{R}$, let

$$\eta_{\mathrm{l}}(x) = \inf\{t \geq 0 : B_t \leq -x\}, \qquad \eta_{\mathrm{r}}(x) = \sup\{t \geq 0 : \Pi_t = x\}.$$

We fix $\varepsilon > 0$ and $r \geq 0$ and first introduce some auxiliary events. For $A > 0$, define

$$\mathcal{E}^1(A) = \left\{ \begin{array}{lll} \min_{[0,A]} \gamma < -6r, & \min_{[A,A^2]} \gamma < -6r, & \min_{[A^2,A^3]} \gamma < -6r, \\ \min_{[-A,0]} \gamma < -6r, & \min_{[-A^2,-A]} \gamma < -6r, & \min_{[-A^3,-A^2]} \gamma < -6r \end{array} \right\}.$$

Next, for $u_0 > 0$, $A > 0$, let

$$\mathcal{E}^2(A, u_0) = \{\eta_{\mathrm{l}}(A^3) \leq u_0\}, \qquad \mathcal{E}^3(A, u_0) = \{\eta_{\mathrm{r}}(A^3) \leq u_0\}.$$

For $u_2 \geq u_1 > 0$, let

$$\mathcal{E}^4(u_1, u_2) = \Big\{ \inf_{[u_2,\infty)} \Pi > \min_{[u_1,u_2]} \Pi \Big\}.$$

For $u_3 \geq u_2 > 0$ and $T \geq u_3$, let

$$\mathcal{E}^5(u_2, u_3, T) = \Big\{ \min_{[0,T-u_3]} F > \min_{[T-u_3,T-u_2]} F \Big\}.$$

Standard properties of Brownian motion imply that there exist $A > 0$ such that $\mathbb{P}(\mathcal{E}^1) \geq 1 - \varepsilon/10$, and we fix A accordingly. Then we can find $u_0 > 0$ in such a way that $\mathbb{P}(\mathcal{E}^2) \geq 1 - \varepsilon/10$ and $\mathbb{P}(\mathcal{E}^3) \geq 1 - \varepsilon/10$, due to the fact that Π is transient. Then we can find u_1 and u_2 with $u_2 \geq u_1 \geq u_0$ such that $\mathbb{P}(\mathcal{E}^4) \geq 1 - \varepsilon/10$.

At last, we claim that we can find u_3 satisfying $u_3 \geq u_2$ and T_0' with $T_0' \geq 2u_3$ such that for $T \geq T_0'$, $\mathbb{P}(\mathcal{E}^5) \geq 1 - \varepsilon/10$. To see this, let $A' > 0$ be a number whose exact value will be fixed later on. If T is such that $\sigma(T) > A'$, note that $\mathcal{E}^5(u_2, u_3, T)$ contains the event $\{\tau_{\sigma(T)-A'} \in [T-u_3, T-u_2]\}$, where τ_x is the first hitting time of $-x$ by the first-passage bridge F. A use of (6.8) and the strong Markov property shows that the law of $T - \tau_{\sigma(T)-A'}$ has a density given by

$$g_t(A') \cdot \frac{g_{T-t}(\sigma(T) - A')}{g_T(\sigma(T))}, \qquad 0 < t < T.$$

It is a simple exercise to see that this converges to $g_t^{(\theta)}(A') = g_t(A')\exp(\theta A' - \theta^2 t/2)$ as $T \to \infty$, which in turn is the probability density of the first hitting time of level $-A'$ by a Brownian motion with drift $-\theta$. An application of Scheffé's lemma entails that $\mathbb{P}(T - \tau_{\sigma(T)-A'} \in [u_2, u_3])$ converges to $\int_{u_2}^{u_3} g_t^{(\theta)}(A')\,\mathrm{d}t$. For a fixed u_2, one can choose A' such that $\int_{u_2}^{\infty} g_t^{(\theta)}(A')\,\mathrm{d}t \geq 1 - \varepsilon/40$ (this is clear from the interpretation of $g_t^{\theta}(A')$ as the density of the hitting time of $-A'$ for a Brownian motion with drift). For this choice of A', we can then choose u_3 such that $\int_{u_2}^{u_3} g_t^{(\theta)}(A')\,\mathrm{d}t \geq 1 - \varepsilon/20$. Finally, we see that for T large enough, one has

$$\mathbb{P}\big(\mathcal{E}^5(u_2, u_3, T)\big) \geq \mathbb{P}\big(T - \tau_{\sigma(T)-A'} \in [u_2, u_3]\big) \geq 1 - \varepsilon/10,$$

as wanted.

We now fix numbers $A, u_3 \geq u_2 \geq u_1 \geq u_0$ and T_0' as specified above. By Lemmas 6.4 and 6.5, we deduce that we can find $T_0 > T_0'$ in such a way that for every $T \geq T_0$, the processes F, b, B, Π, γ can be coupled on the same probability space such that the event

$$\mathcal{E}^6(T) = \left\{ \begin{array}{lll} F_t = B_t, & F_{T-t} = \Pi_t - \sigma(T) & \text{for } t \in [0, u_3], \\ b_x = \gamma_x, & b_{\sigma(T)-x} = \gamma_{-x} & \text{for } x \in [0, A^3] \end{array} \right\}$$

has probability at least $1 - \varepsilon/2$, F is independent of b, B is independent of Π, and γ is independent of (B, Π).

Now fix $T \geq T_0$. Recall that the label function $(W_t, 0 \leq t \leq T)$ of the Brownian disk $\mathsf{BD}_{T,\sigma(T)}$ is defined in terms of F, b, and the snake Z; see Section 6.2.1. We put

$$\mathbb{W}_t = \begin{cases} W_t & \text{if } t \in [0, u_1], \\ W_{T+t} & \text{if } t \in [-u_1, 0]. \end{cases}$$

Given F, the process $(\mathbb{W}_t)_{t \in [-u_1, u_1]}$ is Gaussian; moreover, if we restrict ourselves to the event $\mathcal{E}^5$, we have

$$\underline{F}_{T-t} = \min_{[T-u_3, T-t]} F \qquad \text{for } t \in [0, u_1].$$

Hence the covariance of $(\mathbb{W}_t)_{t \in [-u_1, u_1]}$ is on $\mathcal{E}^5$ a function of the process

$$\big((F_t)_{0 \leq t \leq u_3}, (F_{T-t})_{0 \leq t \leq u_3}\big). \tag{6.15}$$

We turn to the Brownian half-plane and its label function $W^\theta = (W_t^\theta, t \in \mathbb{R})$, which are defined in terms of B, Π, γ, Z^θ; see again Section 6.2.1. Conditionally on (B, Π), W^θ is a Gaussian process. Moreover, since on the event $\mathcal{E}^4$,

$$\inf_{[t,\infty)} \Pi = \min_{[t, u_3]} \Pi \qquad \text{for } t \in [0, u_1],$$

the covariance of the restriction of W^θ to $[-u_1, u_1]$ is on $\mathcal{E}^4$ given by exactly the same function of the process $((B_t)_{0 \leq t \leq u_3}, (\Pi_t)_{0 \leq t \leq u_3})$ as the covariance of

$(\mathbb{W}_t)_{t\in[-u_1,u_1]}$ as a function of the process in (6.15). Since a shift of Π does not affect the covariance, we can instead consider the process

$$\big((B_t)_{0\le t\le u_3}, \big(\Pi_t - \sigma(T)\big)_{0\le t\le u_3}\big). \tag{6.16}$$

On the event $\mathcal{E}^6$, both processes (6.15) and (6.16) coincide. On the event $\mathcal{E}^4\cap\mathcal{E}^5\cap\mathcal{E}^6$, we can therefore construct W and W^θ in such a way that

$$W_t = W^\theta_t, \qquad W_{T-t} = W^\theta_{-t} \qquad \text{for all } t\in[0,u_1]. \tag{6.17}$$

We shall now work on the event $\mathcal{F} = \mathcal{E}^1\cap\mathcal{E}^2\cap\mathcal{E}^3\cap\mathcal{E}^4\cap\mathcal{E}^5\cap\mathcal{E}^6$, which has probability at least $1-\varepsilon$, and assume that the identity (6.17) holds true. According to our convention explained in Remark 6.3, we mostly drop T from the notation.

We follow a strategy similar to [20]. Let $s,t\in[0,T]$. If either both $s,t\in[0,T/2]$ or both $s,t\in[T/2,T]$, we let

$$\tilde{d}_W(s,t) = W_s + W_t - 2\min_{[s\wedge t, s\vee t]} W.$$

Otherwise, we set

$$\tilde{d}_W(s,t) = W_s + W_t - 2\min_{[0,s\wedge t]\cup[s\vee t,T]} W.$$

In the notation from above, we have the following.

LEMMA 6.7. *Assume $\mathcal{F}$ holds.*

(a) *For every $t\in[\eta_{\mathrm{l}}(A), T-\eta_{\mathrm{r}}(A)]$, $D(0,t) > r$.*

(b) *For every $s,t\in[0,\eta_{\mathrm{l}}(A)]\cup[0,T-\eta_{\mathrm{r}}(A)]$ with $\max\{D(0,s),D(0,t)\}\le r$, it holds that*

$$D(s,t) = \inf_{s_1,t_1,\dots,s_k,t_k}\sum_{i=1}^k \tilde{d}_W(s_i,t_i), \tag{6.18}$$

where the infimum is over all possible choices of $k\in\mathbb{N}$ and reals $s_1,\dots,s_k,t_1,\dots,t_k\in[0,\eta_{\mathrm{l}}(A^2)]\cup[T-\eta_{\mathrm{r}}(A^2),T]$ such that $s_1=s$, $t_k=t$, and $d_F(t_i,s_{i+1})=0$ for $1\le i\le k-1$.

PROOF. (a) If $t\in[\eta_{\mathrm{l}}(A), T-\eta_{\mathrm{r}}(A)]$, then the cactus bound (6.13) yields

$$D(0,t) \ge W_t - 2\max\Big\{\min_{[\![0,t]\!]_{\mathcal{T}_F}} W, \min_{[\![t,0]\!]_{\mathcal{T}_F}} W\Big\}. \tag{6.19}$$

Let us show how to bound the first minimum on the right-hand side. On the event $\mathcal{E}^6$, $b_x=\gamma_x$ for $x\in[0,A^3]$ and $F_t = B_t$ for $t\in[0,u_3]$. On the event $\mathcal{E}^2$, we know that the first instant $\eta_{\mathrm{l}}(A)$ when B attains the value $-A$ is bounded from above by u_0, which satisfies $u_0\le u_1\le u_3$. It follows that for $t\ge\eta_{\mathrm{l}}(A)$, the geodesic segment in $\mathcal{T}_F$ between $p_F(0)$ and $p_F(t)$ contains the segment $[\![0,\eta_{\mathrm{l}}(A)]\!]_{\mathcal{T}_F}$ and, therefore, all the vertices of the form $p_F(\eta_l(x))$ for $0 \le x \le A$. Moreover, on $\mathcal{E}^1$, $\min_{[0,A]} \gamma < -6r$. Going back to the definition of W (and using the fact that Z_t equals zero if $\underline{F}$ attains a new minimum at t), we obtain that the first minimum on the right-hand side is bounded from above by $-6r$.

For the second minimum on the right of (6.19), we first observe that on the event $\mathcal{E}^6$, we have also $b_{L-x} = \gamma_{-x}$ for $x \in [0, A^3]$ and $F_{T-t} = \Pi_t - \sigma(T)$ for $t \in [0, u_3]$. Now on $\mathcal{E}^5 \cap \mathcal{E}^6$, we have

$$\underline{F}_{T-t} = \min_{[T-u_3,T-t]} F = \min_{[t,u_3]} \bigl(\Pi - \sigma(T)\bigr) \qquad \text{for } t \in [0, u_1].$$

But on $\mathcal{E}^3$, $\eta_r(A) \le u_0 \le u_1$, so that in particular

$$\underline{F}_{T-\eta_r(A)} = \min_{[\eta_r(A),u_3]} \bigl(\Pi - \sigma(T)\bigr) \ge A - \sigma(T),$$

where for the last inequality we used the fact that $\Pi_t \ge A$ for $t \ge \eta_r(A)$. On $\mathcal{E}^1$, also $\min_{[-A,0]} \gamma < -6r$, and since $[\![t, 0]\!]_{\mathcal{T}_F}$ contains all the vertices of the form $p_F(T - \eta_r(x))$ for $0 \le x \le A$, the second minimum is bounded above again by $-6r$. This proves $D(0,t) \ge 6r$ whenever $t \in [\eta_l(A), T - \eta_r(A)]$, which is more than we claimed.

(b) Recall that $D(s,t)$ is given by

$$\inf\Biggl\{\sum_{i=1}^k d_W(s_i,t_i) : k \ge 1, s_1, \dots, s_k, t_1, \dots, t_k \in [0,T], s_1 = s, t_k = t,$$

$$d_F(t_i, s_{i+1}) = 0 \text{ for every } i \in \{1, \dots, k-1\}\Biggr\}. \tag{6.20}$$

Since $D(s,t) \le D(0,s) + D(0,t) \le 2r$ for s,t as in the statement, it suffices to look at $s_1, \dots, s_k, t_1, \dots, t_k \in [0,T]$ with

$$\sum_{i=1}^k d_W(s_i,t_i) \le 3r. \tag{6.21}$$

We now argue that on the right-hand side of (6.20), we can restrict ourselves to reals $s_1, \dots, s_k, t_1, \dots, t_k \in [0, \eta_l(A^2)] \cup [T - \eta_r(A^2), T]$. Note that from the cactus bound and the fact that $W_0 = W_T = 0$, we have $|W_s| \le r$ whenever $D(0,s) \le r$. Therefore, the cactus bound gives

$$D(s,t_i) \ge -r - \max\Bigl\{\min_{[\![s,t_i]\!]_{\mathcal{T}_F}} W, \min_{[\![t_i,s]\!]_{\mathcal{T}_F}} W\Bigr\}.$$

On the event $\mathcal{E}^6$, $b_x = \gamma_x$, $b_{L-x} = \gamma_{-x}$ for $x \in [0, A^3]$, and $F_t = B_t$, $F_{T-t} = \Pi_t - \sigma(T)$ for $t \in [0, u_3]$. Moreover, on $\mathcal{E}^2$, we have $\eta_l(A^2) \le u_0$, $\eta_r(A^2) \le u_0$.

Recall that by assumption $s \in [0, \eta_l(A)] \cup [T - \eta_r(A), T]$. If there is $i \in \{1, \dots, k\}$ such that t_i is not included in $[0, \eta_l(A^2)] \cup [T - \eta_r(A^2), T]$, then both minima on the right-hand side in the display above are taken over segments which include either $[\![\eta_l(A), \eta_l(A^2)]\!]_{\mathcal{T}_F}$ or $[\![T - \eta_r(A^2), T - \eta_r(A)]\!]_{\mathcal{T}_F}$. Therefore, in this case,

$$D(s,t_i) \geq -r - \max\Big\{ \min_{[\![\eta_l(A),\eta_l(A^2)]\!]_{\mathcal{T}_F}} W, \min_{[\![T-\eta_r(A^2),T-\eta_r(A)]\!]_{\mathcal{T}_F}} W \Big\}. \tag{6.22}$$

We can now argue similar to (a) to show that both minima are bounded from above by $-6r$. Since $[-A^2, -A] \subset B([\eta_l(A), \eta_l(A^2)])$, and since the geodesic segment $[\![\eta_l(A), \eta_l(A^2)]\!]_{\mathcal{T}_F}$ contains all the vertices of the form $\eta_l(x)$ for $A \leq x \leq A^2$, the bound on the first minimum follows from the fact that on $\mathcal{E}^1$, $\min_{[A,A^2]} \gamma < -6r$. The second minimum is treated similarly and left to the reader.

With these bounds, we obtain $D(s,t_i) \geq 5r$. On the other hand, we know from (6.21) that $D(s,t_i) \leq 3r$, a contradiction. The case where s_i is not included in $[0, \eta_l(A^2)] \cup [T - \eta_r(A^2), T]$ for some $i \in \{1,\dots,k\}$ is analogous.

Therefore, we can restrict ourselves in (6.20) to reals $s_1,\dots,s_k,t_1,\dots,t_k \in [0,\eta_l(A^2)] \cup [T - \eta_r(A^2), T]$. We still have to show that we can replace d_W in (6.20) by $\tilde{d}_W$. Let $s_1,\dots,s_k,t_1,\dots,t_k \in [0,\eta_l(A^2)] \cup [T - \eta_r(A^2), T]$ with $s_1 = s$, $t_k = t$ and such that (6.21) holds. Assume first that there is $i \in \{1,\dots,k\}$ such that $s_i \in [0, \eta_l(A^2)]$ and $t_i \in [T - \eta_r(A^2), T]$, and let us show that then $d_W(s_i,t_i) = \tilde{d}_W(s_i,t_i)$. First, by (6.21) in the first inequality,

$$3r \geq d_W(s, s_i) \geq W_s - W_{s_i}.$$

Since $W_s \geq -r$, this shows $W_{s_i} \geq -4r$, and identically one obtains $W_{t_i} \geq -4r$. Using again (6.21),

$$\begin{aligned} 3r &\geq d_W(s_i,t_i) \\ &= W_{s_i} + W_{t_i} - 2\max\Big\{ \min_{[s_i,t_i]} W, \min_{[0,s_i]\cup[t_i,T]} W \Big\} \\ &\geq -8r - 2\max\Big\{ \min_{[s_i,t_i]} W, \min_{[0,s_i]\cup[t_i,T]} W \Big\}. \end{aligned}$$

We claim that this last inequality can only hold if the maximum is attained at the second minimum (which means precisely $d_W(s_i,t_i) = \tilde{d}_W(s_i,t_i)$). Indeed, if $s_i \in [0,\eta_l(A^2)]$ and $t_i \in [T - \eta_r(A^2), T]$, then $[s_i,t_i]$ contains the interval $[\eta_l(A^2), \eta_l(A^3)]$. Arguing in the same way as for the first minimum in (6.22), we deduce that $\min_{[s_i,t_i]} W \leq -6r$, which proves our claim.

The case where $t_i \in [0,\eta_l(A^2)]$ and $s_i \in [T - \eta_r(A^2), T]$ is treated by symmetry. Assume now both s_i, t_i lie in $[0, \eta_l(A^2)]$. Then the interval $[s_i \vee t_i, T]$ contains the interval $[\eta_l(A^2), \eta_l(A^3)]$, so that $\min_{[s_i \vee t_i, T]} W \leq -6r$ by the same reasoning, which gives again $d_W(s_i,t_i) = \tilde{d}_W(s_i,t_i)$. If both s_i, t_i lie in $[T - \eta_r(A^2), T]$, then $[0, s_i \wedge t_i]$ contains $[T - \eta_r(A^3), T - \eta_r(A^2)]$, and the minimum of W over this interval is again bounded from above by $-6r$, using arguments as for the second minimum in (6.19) (or (6.22)). This leads to $d_W(s_i, t_i) = \tilde{d}_W(s_i, t_i)$ also in this case, which completes the proof of (b). □

We turn to the analogous statement for the pseudo-distance function D_θ of the Brownian half-plane BHP_θ. Recall the definition of (X^θ, W^θ); cf. Section 6.2.1.

LEMMA 6.8. *Assume $\mathcal{F}$ holds.*

(a) *For every $t' \in (-\infty, -\eta_{\mathrm{r}}(A)] \cup [\eta_{\mathrm{l}}(A), \infty)$, $D_\theta(0, t') > r$.*

(b) *For every $s', t' \in [-\eta_{\mathrm{r}}(A), \eta_{\mathrm{l}}(A)]$ with $\max\{D_\theta(0, s'), D_\theta(0, t')\} \le r$, it holds that*

$$D_\theta(s', t') = \inf_{s_1', t_1', \dots, s_k', t_k'} \sum_{i=1}^{k} d_{W^\theta}(s_i', t_i'), \tag{6.23}$$

where the infimum is over all possible choices of $k \in \mathbb{N}$ and reals $s_1', \dots, s_k', t_1', \dots, t_k' \in [-\eta_{\mathrm{r}}(A^2), \eta_{\mathrm{l}}(A^2)]$ such that $s_1' = s'$, $t_k' = t'$ and $d_{X^\theta}(t_i', s_{i+1}') = 0$ for $1 \le i \le k-1$.

PROOF. Essentially, one can rely on the identity (6.17) and then follow the proof of Lemma 6.7. Let us now sketch how to prove (a); the proof of (b) is left to the reader. If $t' \in (-\infty, -\eta_{\mathrm{r}}(A)]$, the cactus bound (6.14) gives

$$D_\theta(0, t') \ge - \min_{[\![-\eta_{\mathrm{r}}(A), 0]\!]_{\mathcal{T}_{X^\theta}}} W^\theta.$$

The very definitions of W^θ and $\eta_{\mathrm{r}}(A)$ together with the fact that on $\mathcal{E}^1$, $\min_{[-A,0]} \gamma < -6r$, entail that the minimum is bounded from above by $-6r$. The same bound holds if $t' \in [\eta_{\mathrm{l}}(A), \infty)$, which proves (a). □

Combining Lemmas 6.7 and 6.8, we obtain the following corollary. For the rest of the proof of Proposition 6.6, we set for $u \in [0, T]$,

$$I(u) = \begin{cases} u & \text{if } u \in [0, T/2], \\ u - T & \text{if } u \in [T/2, T]. \end{cases}$$

COROLLARY 6.9. *Assume $\mathcal{F}$ holds. Let $s, t \in [0, \eta_{\mathrm{l}}(A)] \cup [T - \eta_{\mathrm{r}}(A), T]$. Then $\max\{D(0, s), D(0, t)\} \le r$ if and only if $\max\{D_\theta(0, I(s)), D_\theta(0, I(t))\} \le r$. Under these conditions,*

$$D(s, t) = D_\theta\big(I(s), I(t)\big).$$

PROOF. Let $s, t \in [0, \eta_{\mathrm{l}}(A)] \cup [T - \eta_{\mathrm{r}}(A), T]$. We first claim that the expression on the right-hand side of formula (6.18) agrees with that on the right-hand side of (6.23) for $s' = I(s)$, $t' = I(t)$. First, recall that on $\mathcal{F}$, $\max\{\eta_{\mathrm{l}}(A^2), \eta_{\mathrm{r}}(A^2)\} \le$

$u_0 \le T/2$. Therefore, $u \in [0, \eta_l(A^2)] \cup [T - \eta_r(A^2), T]$ if and only if $I(u) \in [-\eta_r(A^2), \eta_l(A^2)]$. Now let $s_1, \dots, s_k, t_1, \dots, t_k \in [0, \eta_l(A^2)] \cup [T - \eta_r(A^2), T]$ such that $s_1 = s$ and $t_k = t$. On $\mathcal{F}$, we have $d_F(t_i, s_{i+1}) = 0$ if and only if $d_{X^\theta}(I(t_i), I(s_{i+1})) = 0$, and $\tilde{d}_W(s_i, t_i) = d_{W^\theta}(I(s_i), I(t_i))$ for each $i \in \{1, \dots, k\}$, which proves our claim. Next, we argue that $\max\{D(0, s), D(0, t)\} \le r$ implies $\max\{D_\theta(0, I(s)), D_\theta(0, I(t))\} \le r$. Indeed, the right-hand side of (6.23) specialized to $s' = I(s)$, $t' = 0$ yields an upper bound on $D_\theta(0, I(s))$, and then the equality of the right-hand sides of (6.18) and (6.23) just shown gives $D_\theta(0, I(s)) \le D(0, s) \le r$, and the same for $D_\theta(0, I(t))$. The fact that $\max\{D_\theta(0, I(s)), D_\theta(0, I(t))\} \le r$ implies $\max\{D(0, s), D(0, t)\} \le r$ follows from a symmetric argument.

Under these conditions, Lemma 6.7 shows that the right-hand side of (6.18) is equal to $D(s, t)$, whereas by Lemma 6.8, the right-hand side of (6.23) is equal to $D_\theta(I(s), I(t))$. (Note here that on $\mathcal{F}$, $s, t \in [0, \eta_l(A)] \cup [T - \eta_r(A), T]$ is clearly equivalent to $I(s), I(t) \in [-\eta_r(A), \eta_l(A)]$.) Using once more the equality of the right-hand sides of (6.18) and (6.23), we obtain $D(s, t) = D_\theta(I(s), I(t))$, and the proof of the corollary is complete. □

We complete the proof of Proposition 6.6 by showing that the balls $B_r(\mathsf{BD}_{T,\sigma(T)})$ and $B_r(\mathsf{BHP}_\theta)$ are isometric on the event $\mathcal{F}$. By Lemma 6.7(a), points in $B_r(\mathsf{BD}_{T,\sigma(T)})$ are on $\mathcal{F}$ equivalence classes of the form $[s]$ for $s \in [-\eta_r(A), \eta_l(A)]$. By the last statement of Corollary 6.9, we deduce that the map I from above can be viewed as an isometric map from $B_r(\mathsf{BD}_{T,\sigma(T)})$ to the quotient $\mathsf{BHP}_\theta = (\mathbb{R}/\{D_\theta = 0\}, D_\theta, \rho_\theta)$. From Lemma 6.8(a), we see that I maps $B_r(\mathsf{BD}_{T,\sigma(T)})$ onto $B_r(\mathsf{BHP}_\theta)$, and it sends ρ, the equivalence class of 0 in $\mathsf{BD}_{T,\sigma(T)}$, to ρ_θ, the equivalence class of 0 in BHP_θ. This completes the proof of the proposition. □

We end this section by improving Proposition 6.6 to the statement of Theorem 3.7.

6.2.5. *Proof of Theorem* 3.7. We will need some known facts about the Brownian disks of finite volume, mostly from Bettinelli [9, 10]. We will simply write $\mathsf{Y} = ([0, T]/\{D = 0\}, D, \rho)$ instead of $\mathsf{BD}_{T,\sigma(T)}$, and, accordingly, $p_{\mathsf{Y}} : [0, T] \to \mathsf{Y}$ denotes the canonical projection.

Although many of our quantities will in the following depend on T, we follow our convention explained in Remark 6.3 and mostly omit T from the notation.

LEMMA 6.10 (Proposition 17 in [10]). *Let $s, t \in [0, T]$ with $s \ne t$ be such that $p_{\mathsf{Y}}(s) = p_{\mathsf{Y}}(t)$ (equivalently $D(s, t) = 0$). Then either $d_F(s, t) = 0$ or $d_W(s, t) = 0$. Moreover, the topology of Y is equal to the quotient topology of $[0, T]/\{D = 0\}$.*

LEMMA 6.11 (Theorem 2 and Proposition 21 in [9]). *Almost surely, the space* Y *is homeomorphic to the closed unit disk* $\overline{\mathbb{D}}$. *The boundary of* Y *as a topological surface is determined by*

$$p_{\mathsf{Y}}^{-1}(\partial \mathsf{Y}) = \{s \in [0, T] : F_s = \underline{F}_s\}.$$

Let f be a real-valued function defined on an interval $J \subset \mathbb{R}$, and let $t \in J$. We say that t is a right-increase point of f if there exists $\varepsilon > 0$ such that $[t, t + \varepsilon] \subset J$ and $f(s) \geq f(t)$ for every $s \in [t, t + \varepsilon]$. Left-increase points are defined similarly, and a unilateral increase point is a time t which is either a left-increase point or a right-increase point. Note for instance that the preceding lemma implies that a point of $\partial\mathsf{Y}$ is necessarily of the form $p_{\mathsf{Y}}(s)$, where s is a unilateral increase point of F.

LEMMA 6.12 (Lemma 12 in [9]). *Almost surely, the sets of unilateral increase points of* F *and* W *are disjoint.*

The following lemma is not strictly needed but useful; see also [32].

LEMMA 6.13 (Lemma 11 in [9]). *Almost surely, there exists a unique* $s_* \in (0, T)$ *such that* $W_{s_*} = \min_{[0,T]} W$.

For $t \in [0, T)$, define

$$\Phi_t(r) = \inf\{s \geq_\circ t : W_s = W_t - r\}, \qquad 0 \leq r \leq W_t - W_{s_*},$$

where in the notation $\geq_\circ$, it should be understood that we consider the cyclic order in $[0, T]$ when T and 0 are identified. More precisely, identifying $[0, T)$ with $\mathbb{R}/T\mathbb{Z}$, for $s, t \in [0, T)$, let $[s, t]_\circ$ be the cyclic interval from s to t, namely, $[s, t]_\circ = [s, t]$ if $s \leq t$ and $[s, t]_\circ = [s, T) \cup [0, t]$ if $t < s$. Then $\Phi_t(r) = s$ if and only if $W_u > W_t - r$ for every $u \in [t, s]_\circ \setminus \{s\}$, and $W_s = W_t - r$. The properties that we will need are summarized in the following statement. For the rest of this section, the time $s_* \in (0, T)$ is specified as in Lemma 6.13.

LEMMA 6.14. *The following properties hold almost surely*:

(a) *For every* $t \in [0, T)$, *the path* $\Gamma_t = p_{\mathsf{Y}} \circ \Phi_t$ *is a geodesic path from* $p_{\mathsf{Y}}(t)$ *to* $x_* = p_{\mathsf{Y}}(s_*)$.

(b) *For every geodesic path* Γ *to* x_* *in* Y, *there exists a unique* $t \in [0, T)$ *such that* $\Gamma_t = \Gamma$.

(c) *For every* $t \in [0, T)$, *the path* Γ_t *intersects* $\partial\mathsf{Y}$, *if at all, only at its origin* $\Gamma_t(0)$.

(d) *Let* $s, t \in [0, T)$ *with* $s \neq t$. *Then the intersection of the sets* $\{\Gamma_s(r) : 0 \leq r \leq D(s_*, s)\}$ *and* $\{\Gamma_t(r) : 0 \leq r \leq D(s_*, t)\}$ *is the set*

$$\Big\{\Gamma_s\big(D(s_*, s) - r\big) : 0 \leq r \leq \max\Big\{\inf_{[s,t]_\circ} W, \inf_{[t,s]_\circ} W\Big\} - W_{s_*}\Big\}.$$

In particular, there exists $\varepsilon > 0$ *such that* $\{\Gamma_s(r) : 0 \leq r \leq \varepsilon\}$ *and* $\{\Gamma_t(r) : 0 \leq r \leq \varepsilon\}$ *are disjoint*.

We note that the length of Γ_t is given by $D(s_*, t) = W_t - W_{s_*}$; see, for example, (2) in [10].

PROOF. (a) and (b) are proved in [10], Proposition 23. To prove (c), we notice that from the definition of Φ_t, every point in the set $\{\Gamma_t(r) : 0 < r \leq D(s_*, t)\}$ must be of the form $p_{\mathsf{Y}}(s)$, where s is a left-increase point of W. By Lemma 6.12, it cannot be a unilateral increase point of F, and thus $p_{\mathsf{Y}}(s)$ is not in $\partial\mathsf{Y}$ by Lemma 6.11.

To prove (d), we first note that whenever $a < \max\{\inf_{[s,t]_\circ} W, \inf_{[t,s]_\circ} W\}$, it must hold that $\inf\{u \geq_\circ s : W_u = a\} = \inf\{u \geq_\circ t : W_u = a\}$, and the fact that $\Gamma_s(D(s_*, s) - r) = \Gamma_t(D(s_*, t) - r)$ for r in the range given in the statement is a simple rewriting of this property and of the fact that $D(s_*, s) = W_s - W_{s_*}$. On the other hand, if $\max\{\inf_{[s,t]_\circ} W, \inf_{[t,s]_\circ} W\} < a \leq W_s \wedge W_t$, then it is simple to see that $s_a = \inf\{u \geq_\circ s : W_u = a\}$ and $t_a = \inf\{u \geq_\circ t : W_u = a\}$ are such that $d_W(s_a, t_a) > 0$, and since both points are left-increase points for W, this implies that $p_{\mathsf{Y}}(s_a) \neq p_{\mathsf{Y}}(t_a)$ by Lemmas 6.12 and 6.10. We leave it to the reader to check that this implies (d). $\square$

Let $a_0 > 0$, which will be fixed later on, and let $O^0_{\mathsf{BD}} = [0, \eta_{\mathrm{l}}(a_0)] \cup [T - \eta_{\mathrm{r}}(a_0), T]$, where

$$\eta_{\mathrm{l}}(a_0) = \inf\{t \geq 0 : F_t \leq a_0\}, \qquad \eta_{\mathrm{r}}(a_0) = T - \sup\big\{t \leq T : F_t \geq -\sigma(T) + a_0\big\}.$$

We stress that later, we will argue on an event $\mathcal{F}$ where the definitions of $\eta_{\mathrm{l}}(a_0), \eta_{\mathrm{r}}(a_0)$ coincide with those given in the proof of Proposition 6.6.

We reason on the event that $s_* \notin O^0_{\mathsf{BD}}$, which will later be granted (with high probability) by the fact that T is bound to go to infinity. For now, we only assume that $\sigma(T) > 2a_0$ so that by Lemma 6.11, the points $x_{\mathrm{l}} = p_{\mathsf{Y}}(\eta_{\mathrm{l}}(a_0))$ and $x_{\mathrm{r}} = p_{\mathsf{Y}}(T - \eta_{\mathrm{r}}(a_0))$ are distinct elements of $\partial\mathsf{Y}$ (outside an event of zero probability). Let $t_* \in O^0_{\mathsf{BD}}$ be such that $W_{t_*} = \min_{O^0_{\mathsf{BD}}} W$ (this defines t_* uniquely a.s., but we are not going to need this fact explicitly). By (d) in Lemma 6.14, together with the fact that $s_* \notin O^0_{\mathsf{BD}}$, the paths $\Gamma_{\eta_{\mathrm{l}}(a_0)}$ and $\Gamma_{T-\eta_{\mathrm{r}}(a_0)}$ are disjoint until they first meet at the point $y_* = p_{\mathsf{Y}}(t_*)$. We let P be the union of the segments of $\Gamma_{\eta_{\mathrm{l}}(a_0)}$ and $\Gamma_{T-\eta_{\mathrm{r}}(a_0)}$ between $x_{\mathrm{l}}, x_{\mathrm{r}}$ and y_*.

LEMMA 6.15. *In the above setting, the set* P *is a simple curve in* Y *from* x_{l} *to* x_{r}, *that intersects* $\partial\mathsf{Y}$ *only at* x_{l} *and* x_{r}. *Letting* O_{BD} *be the connected component of* $\mathsf{Y} \setminus P$ *that contains* $p_{\mathsf{Y}}(0)$, *then* O_{BD} *is a.s. homeomorphic to the closed half-plane* $\overline{\mathbb{H}}$, *and is the interior of the set* $p_{\mathsf{Y}}(O^0_{\mathsf{BD}})$ *in* Y.

PROOF. The fact that P is a simple path follows from the discussion around its definition, and the fact that it intersects the boundary only at its extremities follows at once from Lemma 6.14(c). The fact that O_{BD} is a.s. homeomorphic to $\overline{\mathbb{H}}$ follows from this and the fact that Y is homeomorphic to $\overline{\mathbb{D}}$. It remains to show that O_{BD} is the interior of the set $p_{\mathsf{Y}}(O^0_{\mathsf{BD}})$.

Note that the curve $\beta : x \mapsto p_{\mathsf{Y}}(\inf\{s \in [0, T] : F_s = -x\})$ is a continuous curve from $[0, \sigma(T)]$ to $\partial\mathsf{Y}$ with same starting and ending point, and taking distinct values otherwise. If we view β as defined on the circle $\mathbb{R}/\sigma(T)\mathbb{Z}$, then it realizes a homeomorphism onto $\partial\mathsf{Y}$. In particular, $p_{\mathsf{Y}}(O^0_{\mathsf{BD}})$ contains the segment S of $\partial\mathsf{Y}$ between x_{l} and x_{r} that contains $\rho = p_{\mathsf{Y}}(0)$ (including $x_{\mathrm{l}}, x_{\mathrm{r}}$), while $p_{\mathsf{Y}}([0, T] \setminus O^0_{\mathsf{BD}})$ contains the other segment which is equal to $S' = \partial\mathsf{Y} \setminus S$. For every $s \in [0, T]$, let

$$\Xi_s(r) = \sup\{t \le s : F_t = F_s - r\}, \qquad 0 \le r \le F_s - \underline{F}_s.$$

Then $p_{\mathsf{Y}} \circ \Xi_s$ defines a continuous path in Y from $p_{\mathsf{Y}}(s)$ to the point $\pi(s) = p_{\mathsf{Y}}(\sup\{t \le s : F_t = \underline{F}_t\})$ which is in $\partial\mathsf{Y}$. Moreover, for every $r \in (0, F_s - \underline{F}_s]$, the point $\Xi_s(r)$ is a right-increase point of F, so by Lemma 6.12 it does not belong to $P \setminus \{x_{\mathrm{l}}, x_{\mathrm{r}}\}$, since the latter set contains only points of the form $p_{\mathsf{Y}}(t)$ where t is a unilateral increase point of W. Clearly, $\pi(s) \in S$ if $s \in O^0_{\mathsf{BD}}$, while $\pi(s) \in S'$ otherwise. We have proved that for every $x \in O_{\mathsf{BD}}$, there exists a continuous path from x to $p_{\mathsf{Y}}(0)$ not intersecting P, while for every $x \in \mathsf{Y} \setminus p_{\mathsf{Y}}(O^0_{\mathsf{BD}})$, there exists a continuous path from x to S' not intersecting P. This shows that O_{BD} and $\mathsf{Y} \setminus p_{\mathsf{Y}}(O^0_{\mathsf{BD}})$ are the two connected components of $\mathsf{Y} \setminus P$. $\square$

To complete the proof of Theorem 3.7, fix $r > 0$ and $0 < \varepsilon < 1$. Let a_0 be large enough so that

$$\mathbb{P}\left(\min_{[0,a_0]} \gamma < -2r, \min_{[-a_0,0]} \gamma < -2r\right) \ge 1 - \varepsilon/4.$$

Then we choose $r_0 > r$ such that, with W^θ the label function of BHP_θ,

$$\mathbb{P}\big(\omega\big(W^\theta, [-\eta_{\mathrm{r}}(a_0), \eta_{\mathrm{l}}(a_0)]\big) < r_0/2\big) \ge 1 - \varepsilon/4,$$

where $\omega(f, I) = \sup_I f - \inf_I f$ is the modulus of continuity of f over the set I. We now use the event $\mathcal{F}$ specified in the proof of Proposition 6.6 on which for every $T \ge T_0(\varepsilon/4)$, the balls $B_{r_0}(\mathsf{Y})$ and $B_{r_0}(\mathsf{BHP}_\theta)$ are isometric with probability at least $1 - \varepsilon/4$ (in the definition of $\mathcal{F}$ we have to make sure that A is chosen so that $A > a_0$). We can moreover assume that T_0 is chosen large enough such that for $T \ge T_0$, the probability of $s_* \notin O^0_{\mathsf{BD}}$ is at least $1 - \varepsilon/4$. Then the intersection of $\mathcal{F} \cap \{s_* \notin O^0_{\mathsf{BD}}\}$ with

$$\left\{\min_{[0,a_0]} \gamma < -2r, \min_{[-a_0,0]} \gamma < -2r\right\} \cap \big\{\omega\big(W^\theta, [-\eta_{\mathrm{r}}(a_0), \eta_{\mathrm{l}}(a_0)]\big) < r_0/2\big\}$$

has probability at least $1 - \varepsilon$. On this event we claim that there are the inclusions

$$B_r(\mathsf{Y}) \subset O_{\mathsf{BD}} \subset B_{r_0}(\mathsf{Y}).$$

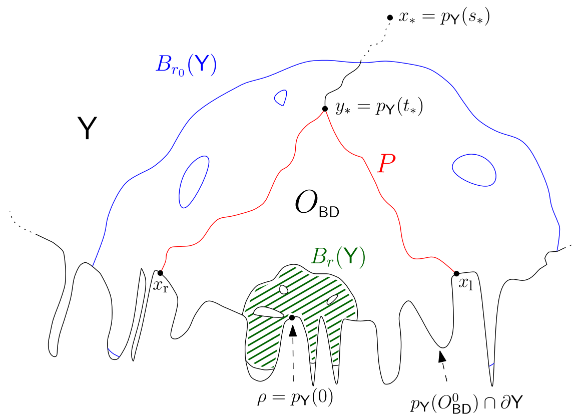

FIG. 9. *The ball $B_r(\mathsf{Y})$ (shaded in green) itself is not simply connected, but it is included in a simply connected open set $O_{\mathsf{BD}} \subset \mathsf{Y}$, which is homeomorphic to the closed half-plane $\overline{\mathbb{H}}$. The set O_{BD} is bordered by the simple curve P (in red) in Y, which is the union of two geodesic segments starting from x_{l} and x_{r}, respectively, and by the boundary segment $S = p_{\mathsf{Y}}(O^0_{\mathsf{BD}}) \cap \partial\mathsf{Y}$. The larger ball $B_{r_0}(\mathsf{Y})$ encompasses O_{BD}.*

The second inclusion comes from the fact that for every $s \in [0, \eta_{\mathrm{l}}(a_0)] \cup [T - \eta_{\mathrm{r}}(a_0), T]$, we have $D(0,s) \le d_W(0,s) \le 2\omega(W^\theta, [-\eta_{\mathrm{r}}(a_0), \eta_{\mathrm{l}}(a_0)])$ (recall that $\mathbb{W} = W^\theta$ on the set $[-\eta_{\mathrm{r}}(A^3), \eta_{\mathrm{l}}(A^3)]$ on $\mathcal{F}$). The first inclusion comes from the cactus bound, with the fact that $\min_{[0,a_0]} \gamma < -2r$ and $\min_{[-a_0,0]} \gamma < -2r$, just as in the proof of (a) in Lemma 6.7. To be more precise, this shows that $B_{2r}(\mathsf{Y}) \subset p_{\mathsf{Y}}([0, \eta_{\mathrm{l}}(a_0)] \cup [T - \eta_{\mathrm{r}}(a_0), T])$, and since O_{BD} is equal to the interior of the latter set, the wanted inclusion follows. We refer to Figure 9 for an illustration.

Finally, recalling that I maps $B_{r_0}(\mathsf{Y})$ isometrically onto $B_{r_0}(\mathsf{BHP}_\theta)$ (see the end of the proof of Proposition 6.6), we deduce that $O_{\mathsf{BHP}} = I(O_{\mathsf{BD}})$ is an open subset of BHP_θ, which concludes the proof of Theorem 3.7.

6.3. *Coupling of quadrangulations of large volumes.* In this section, we provide the proofs of Propositions 3.11 and 3.14. The proof of Proposition 3.11 is in spirit of [20], Lemma 8 and Proposition 9.

PROOF OF PROPOSITION 3.11. Assume $1 \ll \sigma_n \ll n$. Let $\varepsilon > 0$, and set $\vartheta_n = \min\{\sigma_n, n/\sigma_n\}$. Let $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ be uniformly distributed over the set $\mathfrak{F}^n_{\sigma_n} \times \mathfrak{B}_{\sigma_n}$, and consider a triplet $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ of a uniformly labeled critical infinite forest and a uniform infinite bridge.

We first argue that we can find $\delta > 0$ and n_0 such that for all $n \ge n_0$, we can construct $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ on the same probability space such that on an event of probability at least $1 - \varepsilon$, the corresponding balls of radius $2\delta\sqrt{\vartheta_n}$ around the vertices $\mathfrak{f}_n(0)$ and $\mathfrak{f}_\infty(0)$ in the associated quadrangulations are isometric.

For $0 \le k \le \sigma_n - 1$, write $\tau(\mathfrak{f}_n, k)$ for the tree of $\mathfrak{f}_n$ rooted at (k), and put $\tau(\mathfrak{f}_n, \sigma_n) = \tau(\mathfrak{f}_n, 0)$. Similarly, define $\tau(\mathfrak{f}_\infty, k)$ to be the tree of $\mathfrak{f}_\infty$ rooted at (k), where now $k \in \mathbb{Z}$.

As a consequence of Lemmas 5.3 and 5.5, there exist $\delta' > 0$ and $n_0' \in \mathbb{N}$ such that for $n \ge n_0'$, with $A_n = \lfloor \delta' \vartheta_n \rfloor$, we can construct $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ on the same probability space such that if we let

$$\begin{aligned}\mathcal{E}^1(n,\delta') = {} & \{\tau(\mathfrak{f}_n, i) = \tau(\mathfrak{f}_\infty, i), \tau(\mathfrak{f}_n, \sigma_n - i) = \tau(\mathfrak{f}_\infty, -i), 0 \le i \le A_n\} \\ & \cap \{\mathsf{b}_n(i) = \mathsf{b}_\infty(i), \mathsf{b}_n(\sigma_n - i) = \mathsf{b}_\infty(-i), 1 \le i \le A_n\} \\ & \cap \{\mathfrak{l}_n \restriction \tau(\mathfrak{f}_n, i) = \mathfrak{l}_\infty \restriction \tau(\mathfrak{f}_\infty, i), \\ & \quad \mathfrak{l}_n \restriction \tau(\mathfrak{f}_n, \sigma_n - i) = \mathfrak{l}_\infty \restriction \tau(\mathfrak{f}_\infty, -i), 0 \le i \le A_n\},\end{aligned}$$

then $\mathcal{E}^1(n, \delta')$ has probability at least $1 - \varepsilon/3$. We fix such a δ'. For $\delta > 0$ and $n \in \mathbb{N}$, put

$$\begin{aligned}\mathcal{E}^2(n,\delta) = {} & \Big\{\min_{[0,A_n]} \mathsf{b}_\infty < -5\delta\sqrt{\vartheta_n}, \min_{[-A_n,0]} \mathsf{b}_\infty < -5\delta\sqrt{\vartheta_n}\Big\} \\ & \cap \{-\mathsf{b}_\infty(-1) < \delta^{-1}\},\end{aligned}$$

and let

$$\mathcal{E}^3(n,\delta) = \Big\{\min_{[A_n+1,\sigma_n-(A_n+1)]} \mathsf{b}_n < -5\delta\sqrt{\vartheta_n}\Big\}.$$

Donsker's invariance principle applied to $(\mathsf{b}_\infty(i), i \in \mathbb{Z})$ guarantees that we can find $\delta > 0$ such that for all sufficiently large n, $\mathbb{P}(\mathcal{E}^2(n,\delta)) \ge 1 - \varepsilon/3$. Moreover, provided n is large enough and $\delta > 0$ is sufficiently small, Lemma 5.4 ensures that $\mathbb{P}(\mathcal{E}^3(n,\delta)) \ge 1 - \varepsilon/3$. We fix $n_0 \ge n_0'$ and $\delta > 0$ such that for all $n \ge n_0$, the bounds in the last two displays hold.

From now on, we work on the event $\mathcal{E}^1(n,\delta') \cap \mathcal{E}^2(n,\delta) \cap \mathcal{E}^3(n,\delta)$. Let $(Q_n^{\sigma_n}, v^\bullet) = \Phi_n((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $Q_\infty^\infty = \Phi((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ be the quadrangulations constructed from the triplets $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ *via* the Bouttier–Di Francesco–Guitter mapping. We denote by d_n and d_∞ the graph distances on $V(Q_n^{\sigma_n})$ and $V(Q_\infty^\infty)$. We write

$$\mathfrak{f}_n' = (\tau(\mathfrak{f}_n, \sigma_n - A_n), \dots, \tau(\mathfrak{f}_n, \sigma_n - 1), \tau(\mathfrak{f}_n, 0), \dots, \tau(\mathfrak{f}_n, A_n))$$

for the forest obtained from restricting $\mathfrak{f}_n$ to the last A_n and the first $A_n + 1$ trees, and identically

$$\mathfrak{f}_\infty' = (\tau(\mathfrak{f}_\infty, -A_n), \dots, \tau(\mathfrak{f}_\infty, -1), \tau(\mathfrak{f}_\infty, 0), \dots, \tau(\mathfrak{f}_\infty, A_n)).$$

Recall the cactus bounds (4.3) and (4.6) for $Q_n^{\sigma_n}$ and Q_∞^∞, respectively. For vertices $v \in V(\mathfrak{f}_n) \setminus V(\mathfrak{f}_n')$, we obtain, with $(0) = \mathfrak{f}_n(0)$,

$$d_n((0), v) \ge -\max\Big\{\min_{[0,A_n]} \mathsf{b}_n, \min_{[\sigma_n - A_n, \sigma_n - 1]} \mathsf{b}_n\Big\} \ge 5\delta\sqrt{\vartheta_n},$$

and identically, for vertices $v \in V(\mathfrak{f}_\infty) \setminus V(\mathfrak{f}'_\infty)$, writing now (0) for $\mathfrak{f}_\infty(0)$, $d_\infty((0), v) \geq 5\delta\sqrt{\vartheta_n}$. We proceed now similarly to the second part in the proof of [20], Lemma 8. First, if $u \in V(\mathfrak{f}_n)$ is any vertex with $d_n((0), u) \leq 5\delta\sqrt{\vartheta_n} - 1$, then any vertex on a geodesic path from (0) to u in $Q_n^{\sigma_n}$ satisfies the same bound and must therefore belong to one of the trees in $\mathfrak{f}'_n$. From the construction of edges in the Bouttier–Di Francesco–Guitter mapping, we deduce that any edge of $Q_n^{\sigma_n}$ on such a geodesic path corresponds to an edge of Q_∞^∞ with the same endpoints, provided none of these edges in $Q_n^{\sigma_n}$ connect two vertices w and w' such that the set of vertices between w and w' in the cyclic contour order around the forest $\mathfrak{f}_n$ contains the vertices of $\mathfrak{f}_n \setminus \mathfrak{f}'_n$. In other words, one wants to avoid the case where one of the corners in $\tau(\mathfrak{f}_n, 0), \dots, \tau(\mathfrak{f}_n, A_n)$ would have a successor in $\tau(\mathfrak{f}_n, \sigma_n - A_n), \dots, \tau(\mathfrak{f}_n, \sigma_n - 1)$. But on the event $\mathcal{E}^3(n, \delta)$, the set of vertices between w and w' would in particular contain a (root) vertex of $\mathfrak{f}_n \setminus \mathfrak{f}'_n$ with label less than $-5\delta\sqrt{\vartheta_n}$. This would imply that both vertices w and w' of such an edge have a label which is smaller than $-5\delta\sqrt{\vartheta_n}$, too, in contradiction to the fact that $d_n((0), v) \leq 5\delta\sqrt{\vartheta_n} - 1$ for all vertices v on a geodesic between (0) and u. We deduce that if $u \in V(\mathfrak{f}_n)$ satisfies $d_n((0), u) \leq 5\delta\sqrt{\vartheta_n} - 1$, then $d_\infty((0), u) \leq d_n((0), u)$. Since in turn any edge of Q_∞^∞ on a geodesic between (0) and a vertex $u \in V(\mathfrak{f}_\infty)$ with $d_\infty((0), u) \leq 5\delta\sqrt{\vartheta_n} - 1$ does correspond to an edge of $Q_n^{\sigma_n}$ with the same endpoints, we obtain also $d_n((0), u) \leq d_\infty((0), u)$. Therefore, we have that vertices with distance at most $5\delta\sqrt{\vartheta_n} - 1$ from (0) are the same in $Q_n^{\sigma_n}$ and Q_∞^∞. Recall from Section 4.5.4 the notation for the metric balls around the root and (0), respectively. We claim that

$$d_n(u, v) = d_\infty(u, v) \qquad \text{whenever } u, v \in B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n}). \tag{6.24}$$

Indeed, if u, v are vertices in $B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$, then any geodesic connecting u to v in $Q_n^{\sigma_n}$ must lie entirely in $B^{(0)}_{4\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$, and any edge on such a geodesic corresponds to an edge in Q_∞^∞. Since the converse is also true, we obtain (6.24), and with the correspondence of edges between $Q_n^{\sigma_n}$ and Q_∞^∞ alluded to above we deduce that the balls $B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$ and $B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_\infty^\infty)$ are isometric on an event of probability at least $1 - \varepsilon$.

Finally, recall from the Bouttier–Di Francesco–Guitter bijection that the root vertex ρ_n of $Q_n^{\sigma_n}$ is given by $\mathfrak{f}_n(\mathrm{succ}^{-\mathsf{b}_n(\sigma_n)}(0))$, where conditionally on $\mathsf{b}_n(\sigma_n - 1)$, $\mathsf{b}_n(\sigma_n)$ is uniformly distributed on $\{\mathsf{b}_n(\sigma_n - 1) - 1, \dots, 0\}$. Similarly, the root vertex ρ of Q_∞^∞ is given by $\mathfrak{f}_\infty(\mathrm{succ}^{-\mathsf{b}_\infty(\partial)}(0))$, where conditionally on $\mathsf{b}_\infty(-1)$, $\mathsf{b}_\infty(\partial)$ is uniformly distributed on $\{\mathsf{b}_\infty(-1) - 1, \dots, 0\}$. On the event $\mathcal{E}^1(n, \delta') \cap \mathcal{E}^2(n, \delta)$, we can couple $\mathsf{b}_n(\sigma_n)$ and $\mathsf{b}_\infty(\partial)$ such that $\mathsf{b}_n(\sigma_n) = \mathsf{b}_\infty(\partial)$. Moreover, for n large enough, we have on this event $B_{\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n}) \subset B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$ and $B_{\delta\sqrt{\vartheta_n}}(Q_\infty^\infty) \subset B^{(0)}_{2\delta\sqrt{\vartheta_n}}(Q_\infty^\infty)$. Therefore, we have equality of $B_{\delta\sqrt{\vartheta_n}}(Q_n^{\sigma_n})$ and $B_{\delta\sqrt{\vartheta_n}}(Q_\infty^\infty)$ on the event $\mathcal{E}^1(n, \delta') \cap \mathcal{E}^2(n, \delta) \cap \mathcal{E}^3(n, \delta)$. Local convergence of

$Q_n^{\sigma_n}$ toward UIHPQ in the sense of d_{map} is a direct consequence of this, and the proposition is proved. $\square$

We now turn to the proof of Proposition 3.14. We will adopt the notion of [20], Section 4.3.1, concerning pruned (pointed) trees. More precisely, a (finite) pointed tree consists of a pair $\boldsymbol{\tau} = (\tau, \xi)$, where τ is a tree of finite size and ξ is a distinguished vertex of τ. Given such a pointed tree $\boldsymbol{\tau} = (\tau, \xi)$ and h an integer with $0 \le h < |\xi|$, $\mathscr{P}(\boldsymbol{\tau}, h)$ represents the subtree of τ containing all the vertices $u \in V(\tau)$ such that the height of the most recent common ancestor of u and ξ is strictly less than h, together with the ancestor $[\xi]_h$ of ξ at height exactly h. By pointing $\mathscr{P}(\boldsymbol{\tau}, h)$ at $[\xi]_h$, this subtree is itself considered as a pointed tree. If $h \ge |\xi|$, we agree that $\mathscr{P}(\boldsymbol{\tau}, h) = (\{\varnothing\}, \varnothing)$, where $\varnothing$ represents the root vertex of τ. It is straightforward to see that if $\boldsymbol{\tau} = (\tau, \xi)$ is a pointed tree and h and h' are two integers with $h' \ge h \ge 0$, then

$$\mathscr{P}\big((\boldsymbol{\tau}, h'), h\big) = \mathscr{P}(\boldsymbol{\tau}, h). \tag{6.25}$$

PROOF OF PROPOSITION 3.14. We assume $1 \ll \sigma_n \ll \sqrt{n}$ and fix $\varepsilon > 0$ and $r > 0$ for the rest of this proof. We let $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ be uniformly distributed over the set $\mathfrak{F}_{\sigma_n}^n \times \mathfrak{B}_{\sigma_n}$, and for a given $R \in \mathbb{N}$, we let $((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$ be uniformly distributed over $\mathfrak{F}_{\sigma_n}^{R\sigma_n^2} \times \mathfrak{B}_{\sigma_n}$. Identically to the proof of Proposition 3.11, it suffices to show that we can find $R_0 > 0$ and $n_0 \in \mathbb{N}$ such that for all integers $R \ge R_0$ and all $n \ge n_0$, we can construct $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$ on the same probability space such that on an event of probability at least $1 - \varepsilon$, the corresponding balls of radius $2r\sqrt{\sigma_n}$ around the vertices $\mathfrak{f}_n(0)$ and $\mathfrak{f}'_n(0)$ in the associated quadrangulations are isometric.

For $0 \le k \le \sigma_n - 1$, we let $\tau(\mathfrak{f}_n, k)$ be the tree of $\mathfrak{f}_n$ rooted at (k) and denote by i_* the smallest index such that $|\tau(\mathfrak{f}_n, i_*)| \ge |\tau(\mathfrak{f}_n, k)|$ for all $0 \le k \le \sigma_n - 1$. We shall point the tree $\tau(\mathfrak{f}_n, i_*)$, by choosing conditionally on $\tau(\mathfrak{f}_n, i_*)$ a vertex $\xi_n \in V(\tau(\mathfrak{f}_n, i_*))$ uniformly at random. We write $(\tau(\mathfrak{f}_n, i_*), \xi_n)$ for the pointed tree obtained in this way, and for $h \in \mathbb{N}$, we write $\mathfrak{l}_n \restriction \mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), h)$ for the restriction of the labels $\mathfrak{l}_n$ of $\mathfrak{f}_n$ to the subtree $\mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), h)$ of $(\tau(\mathfrak{f}_n, i_*), \xi_n)$ pruned at height h; see the notation above. Finally, we let (τ_i, ℓ_i), $0 \le i \le \sigma_n - 1$, be a sequence of independent uniformly labeled critical geometric Galton–Watson trees.

For $H \in \mathbb{N}$, set $H_n = H\sigma_n$. Recall that the law of $((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$ depends on $R \in \mathbb{N}$. We claim that for each fixed integer $H \in \mathbb{N}$, provided n and R are sufficiently large, we can construct $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$, $((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$, ξ_n, ξ'_n and (τ_i, ℓ_i) for $0 \le i \le \sigma_n - 1$ on the same probability space such that, with i'_* being defined as i_* but in terms of $\mathfrak{f}'_n$, the event

$$\begin{aligned}&\mathcal{E}^1(n, R, H)\\ &\quad = \{i_* = i'_*\} \cap \{\tau(\mathfrak{f}_n, i) = \tau(\mathfrak{f}'_n, i) = \tau_i, 0 \le i \le \sigma_n - 1, i \ne i_*\}\end{aligned}$$

$$\cap \{\mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), H_n) = \mathscr{P}((\tau(\mathfrak{f}'_n, i'_*), \xi'_n), H_n) \neq (\{\varnothing\}, \varnothing)\}$$
$$\cap \{\mathsf{b}_n(i) = \mathsf{b}'_n(i), 0 \le i \le \sigma_n\}$$
$$\cap \{\mathfrak{l}_n \restriction \tau(\mathfrak{f}_n, i) = \mathfrak{l}'_n \restriction \tau(\mathfrak{f}'_n, i) = \ell_i, 0 \le i \le \sigma_n - 1, i \neq i_*\}$$
$$\cap \{\mathfrak{l}_n \restriction \mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), H_n) = \mathfrak{l}'_n \restriction \mathscr{P}((\tau(\mathfrak{f}'_n, i'_*), \xi'_n), H_n)\}$$

has probability at least $1 - \varepsilon/2$. Let us look separately at the different sets on the right-hand side. First, from Lemma 5.1 we know that $\mathfrak{f}_n$ has with high probability a unique largest tree of order σ_n^2, and its index is uniform in $\{0, \dots, \sigma_n - 1\}$. Moreover, Lemma 5.2 asserts that the other trees of $\mathfrak{f}_n$ are close in total variation to $\sigma_n - 1$ critical geometric Galton–Watson trees. The same holds for $\mathfrak{f}'_n$, from which we deduce that $\mathfrak{f}_n, \mathfrak{f}'_n$ and τ_i, $0 \le i \le \sigma_n - 1$, can be coupled such that the intersection of the first two events on the right-hand side has probability at least $1 - \varepsilon/3$, say. For the event on the second line concerning the pruned trees, we use that fact that conditionally on $|\tau(\mathfrak{f}_n, i_*)| = m_n$, $(\tau(\mathfrak{f}_n, i_*), \xi_n)$ is uniformly distributed over the set of all pointed trees of size m_n. A similar statement holds for $\tau(\mathfrak{f}'_n, i'_*)$. Now by Lemma 5.1, for any $K > 0$, the probability that $|\tau(\mathfrak{f}_n, i_*)| \ge K\sigma_n^2$ tends to one with increasing n, since $n \gg \sigma_n^2$. Similarly, for any given $K > 0$, by choosing R large enough, we can ensure that $|\tau(\mathfrak{f}'_n, i'_*)| \ge K\sigma_n^2$ holds with a probability as close to one as we wish for large n. An application of Proposition 7 of [20] therefore shows that both $\mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), H_n)$ and $\mathscr{P}((\tau(\mathfrak{f}'_n, i'_*), \xi'_n), H_n)$ are for large R and n close in total variation to the so-called uniform infinite tree (or Kesten's tree) pruned at height H_n. Applying the triangle inequality, we see that the total variation distance between $\mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), H_n)$ and $\mathscr{P}((\tau(\mathfrak{f}'_n, i'_*), \xi'_n), H_n)$ can be made as small as we wish, provided R and n are taken sufficiently large.

Combining the above coupling with this last observation, we infer that we can in fact couple $\mathfrak{f}_n$, $\mathfrak{f}'_n$, ξ_n, ξ'_n and τ_i for $0 \le i \le \sigma_n - 1$ such that the intersection of the first three events on the right-hand side has probability at least $1 - \varepsilon/2$ for large R and n. Since the bridges b_n and b'_n have both the same law and are independent of the trees, we can additionally assume that the probability space carries realizations of b_n and b'_n such that $\mathsf{b}_n \equiv \mathsf{b}'_n$. A similar argument allows us to couple the labelings $\mathfrak{l}_n$, $\mathfrak{l}'_n$ and ℓ_i such that the last two events on the right-hand side in the definition of $\mathcal{E}^1(n, R, H)$ hold true. This proves the claim about $\mathcal{E}^1(n, R, H)$.

We will now work on the event $\mathcal{E}^1(n, R, H)$. Let $(Q_n^{\sigma_n}, v^\bullet) = \Phi_n((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $(Q_{R\sigma_n^2}^{\sigma_n}, w^\bullet) = \Phi_{R\sigma_n^2}((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$ be the quadrangulations constructed from $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n)$ and $((\mathfrak{f}'_n, \mathfrak{l}'_n), \mathsf{b}'_n)$, respectively. Recall that $[\xi_n]_{H_n}$ denotes the ancestor of ξ_n in $\tau(\mathfrak{f}_n, i_*)$ at height H_n. Let

$$M_n = -\min_{[\![\varnothing, [\xi_n]_{H_n}]\!]} \mathfrak{l}_n,$$

where $[\![\varnothing, [\xi_n]_{H_n}]\!]$ is the vertex set of the unique injective path in $\tau(\mathfrak{f}_n, i_*)$ connecting the (tree) root $\varnothing$ to $[\xi_n]_{H_n}$. By definition of the labeling $\mathfrak{l}_n$, conditionally on the tree, M_n has the law of the maximum attained by a random walk started at zero and stopped after H_n many steps, with increments uniformly distributed in $\{-1, 0, 1\}$. Setting

$$\mathcal{E}^2(n, H) = \{M_n \geq 5r\sqrt{\sigma_n}\},$$

we can ensure by an application of Donsker's invariance principle that for $H \in \mathbb{N}$ sufficiently large (recall that r was fixed at the beginning, and $H_n = H\sigma_n$), the event $\mathcal{E}^2(n, H)$ has probability at least $1 - \varepsilon/2$. In particular, by choosing $H \in \mathbb{N}$ large enough, we obtain that $\mathcal{E}^1(n, R, H) \cap \mathcal{E}^2(n, H)$ has probability at least $1 - \varepsilon$ for all $R, n \in \mathbb{N}$ sufficiently large.

It remains to convince ourselves that on the event $\mathcal{E}^1(n, R, H) \cap \mathcal{E}^2(n, H)$, the balls $B^{(0)}_{2r\sqrt{\sigma_n}}(Q_n^{\sigma_n})$ and $B^{(0)}_{2r\sqrt{\sigma_n}}(Q^{\sigma_n}_{R\sigma_n^2})$ are isometric. Since the arguments are very close to those given in the proofs of Proposition 3.11 above and [20], Lemma 8, we only sketch them. Write $\varnothing = u_0, u_1, \dots, u_{H_n} = [\xi_n]_{H_n}$ for the vertices of the nonbacktracking path connecting $\varnothing$ to $[\xi_n]_{H_n}$ in $\tau(\mathfrak{f}_n, i_*)$. Let $k_n \in \{0, \dots, H_n\}$ such that

$$\mathfrak{l}_n(u_{k_n}) = -\min_{[\![\varnothing, [\xi_n]_{H_n}]\!]} \mathfrak{l}_n.$$

Recall the identification of $V(Q_n^{\sigma_n}) \setminus \{v^\bullet\}$ with $V(\mathfrak{f}_n)$. Denote by d_n the graph distance on $V(Q_n^{\sigma_n})$. If v is a vertex of $\tau(\mathfrak{f}_n, i_*)$ that does not belong to the subtree $\mathscr{P}((\tau(\mathfrak{f}_n, i_*), \xi_n), k_n)$, then the ancestral lines of v and ξ_n coincide at least up to level k_n. In particular, they both contain the vertex u_{k_n}. For such vertices v, we obtain from the cactus bound (4.3) on the event $\mathcal{E}^1(n, R, H) \cap \mathcal{E}^2(n, H)$ the bound

$$d_n\big((0), v\big) \geq 5r\sqrt{\sigma_n},$$

with $(0) = \mathfrak{f}_n(0)$. See [20], Proof of Lemma 8, for the complete argument (note, however, that (0) might be the root of a tree different from $\tau(\mathfrak{f}_n, i_*)$). On $\mathcal{E}^1(n, R, H)$, using additionally (6.25),

$$\mathscr{P}\big(\big(\tau(\mathfrak{f}_n, i_*), \xi_n\big), k_n\big) = \mathscr{P}\big(\big(\tau(\mathfrak{f}'_n, i'_*), \xi'_n\big), k_n\big),$$

and the labelings $\mathfrak{l}_n$ and $\mathfrak{l}'_n$ restricted to the subtrees on the left and right, respectively, agree. Therefore, a similar inequality holds for $Q^{\sigma_n}_{R\sigma_n^2}$, for vertices v' of $\tau(\mathfrak{f}'_n, i'_*)$ which do not belong to the subtree $\mathscr{P}((\tau(\mathfrak{f}'_n, i'_*), \xi'_n), k_n)$. Adapting now the reasoning of [20], Proof of Lemma 8, to our situation (see also the proof of Proposition 3.11 above), we obtain that vertices with distance at most $5r\sqrt{\sigma_n} - 1$ from (0) are the same in $Q_n^{\sigma_n}$ and $Q^{\sigma_n}_{R\sigma_n^2}$ on the event $\mathcal{E}^1(n, R, H) \cap \mathcal{E}^2(n, H)$, and, with d'_n being the graph distance in $Q^{\sigma_n}_{R\sigma_n^2}$,

$$d_n(u, v) = d'_n(u, v) \qquad \text{whenever } u, v \in B^{(0)}_{2r\sqrt{\sigma_n}}\big(Q_n^{\sigma_n}\big).$$

This completes our proof. $\square$

6.4. *Brownian half-plane with zero skewness.* We work in the usual setting from Section 4.5.4. Our proofs of Theorems 3.3 and 3.6 are essentially consequences of the coupling of balls between the Brownian disk $\mathsf{BD}_{T,\sqrt{T}}$ and the Brownian half-plane BHP (Proposition 6.6), of the fundamental convergence

$$\big(V\big(Q_n^{\sigma_n}\big), (8/9)^{-1/4}n^{-1/4}d_{\mathrm{gr}}, \rho_n\big) \xrightarrow[n\to\infty]{(d)} \mathsf{BD}_\sigma = \mathsf{BD}_{1,\sigma} \tag{6.26}$$

proved in [11], Theorem 1, for the regime $\sigma_n \sim \sigma\sqrt{2n}$ when $\sigma \in (0,\infty)$ is a fixed real, and of the coupling between $Q_n^{\sigma_n}$ and the UIHPQ Q_∞^∞ (Proposition 3.11).

PROOF OF THEOREM 3.6. In view of Remark 2.10, the result follows if we show that for every $r \ge 0$ and every sequence of positive reals $a_n \to \infty$,

$$B_r\big(a_n^{-1}\cdot Q_\infty^\infty\big) \xrightarrow[n\to\infty]{(d)} B_r(\mathsf{BHP})$$

in distribution in $\mathbb{K}$. For notational simplicity, we restrict ourselves to the case $r = 1$. Fix $\varepsilon > 0$. By Proposition 6.6, we find $T_0 = T_0(\varepsilon) > 0$ such that for all $T \ge T_0$, we can construct copies of $\mathsf{BD}_{T,\sqrt{T}}$ and BHP on the same probability space such that

$$B_1(\mathsf{BD}_{T,\sqrt{T}}) = B_1(\mathsf{BHP}) \tag{6.27}$$

with probability at least $1-\varepsilon$.

Let $\sigma_n = \lceil\sqrt{2n}\rceil$. By Proposition 3.11, there exists $\delta > 0$ such that for n large enough, we can couple $Q_n^{\sigma_n}$ and Q_∞^∞ on the same probability space such that with probability at least $1-\varepsilon$, $B_{\delta\sqrt{\sigma_n}}(Q_n^{\sigma_n}) = B_{\delta\sqrt{\sigma_n}}(Q_\infty^\infty)$. We can and will assume that $\delta < 2T_0^{-1/4}$. We put $m_n = \lceil\delta^{-4}a_n^4\rceil$. Then $a_n \le \delta\sqrt{\sigma_{m_n}}$. With m_n taking the role of n, the last observation enables us to find a coupling between $Q_{m_n}^{\sigma_{m_n}}$ and Q_∞^∞ on the same probability space such that for large n, we have with probability at least $1-\varepsilon$,

$$B_{a_n}\big(Q_{m_n}^{\sigma_{m_n}}\big) = B_{a_n}\big(Q_\infty^\infty\big). \tag{6.28}$$

Let $F : \mathbb{K} \to \mathbb{R}$ be bounded and continuous, and put $T = \delta^{-4}(8/9)$. Note that $T \ge T_0$. We work with a coupling of $Q_{m_n}^{\sigma_{m_n}}$ and Q_∞^∞ as well as with a coupling of $\mathsf{BD}_{T,\sqrt{T}}$ and BHP such that the properties just mentioned hold. Then

$$\begin{aligned}
&\big|\mathbb{E}\big[F\big(B_1\big(a_n^{-1}\cdot Q_\infty^\infty\big)\big)\big] - \mathbb{E}\big[F\big(B_1(\mathsf{BHP})\big)\big]\big| \\
&\quad\le \big|\mathbb{E}\big[F\big(a_n^{-1}\cdot B_{a_n}\big(Q_\infty^\infty\big)\big) - F\big(a_n^{-1}\cdot B_{a_n}\big(Q_{m_n}^{\sigma_{m_n}}\big)\big)\big]\big| \\
&\qquad + \big|\mathbb{E}\big[F\big(B_1\big(a_n^{-1}\cdot Q_{m_n}^{\sigma_{m_n}}\big)\big)\big] - \mathbb{E}\big[F\big(B_1(\mathsf{BD}_{T,\sqrt{T}})\big)\big]\big| \\
&\qquad + \big|\mathbb{E}\big[F\big(B_1(\mathsf{BD}_{T,\sqrt{T}})\big) - F\big(B_1(\mathsf{BHP})\big)\big]\big|.
\end{aligned}$$

Using the coupling (6.28) for the first and the coupling (6.27) for the third summand on the right-hand side, we see that both of them are bounded from above by

$2\varepsilon \sup|F|$. The second summand converges to zero as $n \to \infty$, using (6.26) and the scaling relation $\mathsf{BD}_{T,\sqrt{T}} =_d T^{1/4}\mathsf{BD}_1$. This concludes the proof of Theorem 3.6. □

PROOF OF THEOREM 3.3. We have to show that when $1 \ll \sigma_n \ll n$, we have for every $r \geq 0$ and any sequence $1 \ll a_n \ll \min\{\sqrt{\sigma_n}, \sqrt{n/\sigma_n}\}$, $B_r(a_n^{-1} \cdot Q_n^{\sigma_n}) \longrightarrow B_r(\mathsf{BHP})$ in distribution in $\mathbb{K}$ as $n \to \infty$. Let $\varepsilon > 0$ and $r \geq 0$. By Proposition 3.11, we can couple $Q_n^{\sigma_n}$ and Q_∞^∞ on the same probability space such that with probability at least $1-\varepsilon$, for $n \geq n_0$, $B_{ra_n}(Q_n^{\sigma_n}) = B_{ra_n}(Q_\infty^\infty)$. Then, for $F : \mathbb{K} \to \mathbb{R}$ bounded and continuous,

$$
\begin{aligned}
&\bigl|\mathbb{E}\bigl[F\bigl(B_r(a_n^{-1} \cdot Q_n^{\sigma_n})\bigr) - F\bigl(B_r(\mathsf{BHP})\bigr)\bigr]\bigr| \\
&\quad \leq \bigl|\mathbb{E}\bigl[F\bigl(a_n^{-1} \cdot B_{ra_n}(Q_n^{\sigma_n})\bigr) - F\bigl(a_n^{-1} \cdot B_{ra_n}(Q_\infty^\infty)\bigr)\bigr]\bigr| \\
&\qquad + \bigl|\mathbb{E}\bigl[F\bigl(B_r(\mathsf{BHP})\bigr) - F\bigl(a_n^{-1} \cdot B_{ra_n}(Q_\infty^\infty)\bigr)\bigr]\bigr|.
\end{aligned}
$$

Under our coupling, the first summand behind the inequality is bounded by $2\varepsilon \sup|F|$ provided $n \geq n_0$. By Theorem 3.6, the second summand converges to zero as $n \to \infty$. □

REMARK 6.16. Notice that in our proofs of the couplings Proposition 6.6 (between BD_σ and BHP) and Proposition 3.11 (between $Q_n^{\sigma_n}$ and UIHPQ), we construct in fact joint couplings of contour functions, label functions and balls in the corresponding metric spaces. As a consequence, the theorems proved in this section can be strengthened in a way we now exemplify based on Theorem 3.6.

Recall that we view the contour and label functions C_∞ and $\mathfrak{L}_\infty$ that specify the UIHPQ Q_∞^∞ as (random) continuous functions from $\mathbb{R}$ to $\mathbb{R}$; cf. Sections 4.5.4 and 4.2. The Brownian half-plane BHP is constructed from the contour and label functions $X^0 = (X_t^0, t \in \mathbb{R})$ and $W^0 = (W_t^0, t \in \mathbb{R})$ specified in Section 6.2.1.

We now claim that for each $r \geq 0$ and any positive sequence $a_n \to \infty$,

(6.29)
$$
\left(\frac{C_\infty((9/4)a_n^4 \cdot)}{(3/2)a_n^2}, \frac{\mathfrak{L}_\infty((9/4)a_n^4 \cdot)}{a_n}, B_r\bigl(a_n^{-1} \cdot Q_\infty^\infty\bigr)\right) \xrightarrow[n\to\infty]{(d)} \bigl(X^0, W^0, B_r(\mathsf{BHP})\bigr)
$$

jointly in the space $\mathcal{C}(\mathbb{R},\mathbb{R}) \times \mathcal{C}(\mathbb{R},\mathbb{R}) \times \mathbb{K}$. The convergence does also hold with $B_r(a_n^{-1} \cdot Q_\infty^\infty)$ replaced by $B_r^{(0)}(a_n^{-1} \cdot Q_\infty^\infty)$.

To see why (6.29) holds, one has to slightly enhance the proof of Theorem 3.6. Since all the necessary arguments were already given, we restrict ourselves to a sketch of proof and leave it to the reader to fill in the details. We assume $r = 1$ for simplicity. Let $T > 0$, denote by (F, W) the contour and label function of $\mathsf{BD}_{T,\sqrt{T}}$, and set $F(-t) = F(T-t) + \sqrt{T}$ and $W(-t) = W(T-t)$ for $t \in [0, T]$. Now fix $K > 0$.

First, the arguments in the proof of Proposition 6.6 show that for $T > K$ large, one can construct a coupling such that with high probability, equality (6.27) holds jointly with an equality of (F, W) and (X^0, W^0) on $[-K, K]^2 \subset [-T, T]^2$.

Second, let $m_n = \lceil \delta^{-4} a_n^4 \rceil$ and $\sigma_{m_n} = \lceil \sqrt{2m_n} \rceil$ be as in the proof of Theorem 3.6. We extend the contour function C_{m_n} of $Q_{m_n}^{\sigma_{m_n}}$ to $t \in [-(2m_n + \sigma_{m_n}), 0]$ by setting $C_{m_n}(t) = C_{m_n}(2m_n + \sigma_{m_n} + t) + \sigma_{m_n}$. Similarly, we extend the label function $\mathfrak{L}_{m_n}$ to $[-(2m_n + \sigma_{m_n}), 0]$, by letting $\mathfrak{L}_{m_n}(t) = \mathfrak{L}_{m_n}(2m_n + \sigma_{m_n} + t)$ for $t \in [-(2m_n + \sigma_{m_n}), -1]$, and then by linear interpolation on $[-1, 0]$ between $\mathfrak{L}_{m_n}(-1)$ and $\mathfrak{L}_{m_n}(0) = 0$.

From the proof of Proposition 3.11, we deduce that for δ small and n large, one can construct a coupling such that with high probability, equality (6.28) holds jointly with an equality of $(C_{m_n}, \mathfrak{L}_{m_n})$ and $(C_\infty, \mathfrak{L}_\infty)$ on $[-Ka_n^4, Ka_n^4]^2$.

Thanks to [9, 11], we already know that the convergence of $a_n^{-1} \cdot Q_{m_n}^{\sigma_{m_n}}$ to $\mathsf{BD}_{T,\sqrt{T}}$ (with $T = \delta^{-4}(8/9)$) holds jointly with the convergence

$$\left(\frac{C_{m_n}((9/4)a_n^4 \cdot)}{(3/2)a_n^2}, \frac{\mathfrak{L}_{m_n}((9/4)a_n^4 \cdot)}{a_n} \right) \xrightarrow[n\to\infty]{(d)} (F, W)$$

in $\mathcal{C}([-T, T], \mathbb{R})^2$. Putting these observations together, (6.29) follows.

We come back to display (6.29) in the proof of Theorem 3.4.

6.5. *Brownian half-plane with nonzero skewness*. Here, we prove Theorem 3.4, which covers the regime $\sqrt{n} \ll \sigma_n \ll n$ when $a_n \sim 2\sqrt{\theta n/3\sigma_n}$ for some $\theta \in (0, \infty)$. The parameter θ measures the skewness of the limiting Brownian half-plane. Note that the regimes where the space BHP corresponding to the choice $\theta = 0$ appears is already treated in Theorem 3.3.

We work in the usual setting introduced in Section 4.5.4; in particular, the pair $(Q_n^{\sigma_n}, v^\bullet)$ consisting of a quadrangulation and a distinguished vertex is uniformly distributed over $\mathcal{Q}_{n,\sigma_n}^\bullet$ and encoded by a triplet $((\mathfrak{f}_n, \mathfrak{l}_n), \mathsf{b}_n) \in \mathfrak{F}_{\sigma_n}^n \times \mathcal{B}_{\sigma_n}$. The associated contour pair is denoted (C_n, L_n), and the corresponding label function takes the form $\mathfrak{L}_n(t) = L_n(t) + \mathsf{b}_n(-\underline{C}_n(t))$, $0 \le t \le 2n + \sigma_n$.

It will be convenient to view both C_n and $\mathfrak{L}_n$ as continuous functions on $\mathbb{R}$. Let $N = 2n + \sigma_n$. We extend C_n first to $[-N, N]$ by $C_n(t) = C_n(N + t) + \sigma_n$ for $t \in [-N, 0]$, and then to all reals t by setting $C_n(t) = C_n((t \vee (-N)) \wedge N)$. Similarly, we let $\mathfrak{L}_n(t) = \mathfrak{L}_n(N + t)$ for $t \in [-N, -1]$, with linear interpolation on $[-1, 0]$ between $\mathfrak{L}_n(-1)$ and 0. Outside $[-N, N]$, we set $\mathfrak{L}_n(t) = \mathfrak{L}_n((t \vee (-N)) \wedge N)$. In this way, we interpret C_n and $\mathfrak{L}_n$ as elements of $\mathcal{C}(\mathbb{R}, \mathbb{R})$. Recall that they completely determine $(Q_n^{\sigma_n}, v^\bullet)$.

IDEA OF THE PROOF. For fixed $r \ge 0$, the ball $B_{ra_n}(Q_n^{\sigma_n})$ is with high probability encoded by the union of the first ca_n^2 and last ca_n^2 trees of $\mathfrak{f}_n$ for some $c = c(r) > 0$, together with their labels and the corresponding bridge values along the floor of $\mathfrak{f}_n$. In Lemma 6.17, we calculate the Radon–Nikodym derivative of the law of these $2ca_n^2$ trees with respect to the law of $2ca_n^2$ independent critical geometric Galton–Watson trees. In this way, we explicitly relate the laws of $B_{ra_n}(Q_n^{\sigma_n})$ and $B_{ra_n}(Q_\infty^\infty)$ to each other. Since we already know that $a_n^{-1} \cdot B_{ra_n}(Q_\infty^\infty)$ converges to $B_r(\mathsf{BHP}_0)$ jointly with its properly rescaled contour and label functions (see Remark 6.16), it remains to identify the limiting Radon–Nikodym derivative, which we find to be the Radon–Nikodym derivative of a (two-sided) Brownian motion with drift $-\theta$ with respect to standard Brownian motion. An application of the Pitman transform then concludes the proof.

Let us now give the details and first introduce some supplementary notation. For this section, given a continuous function $f : \mathbb{R} \to \mathbb{R}$ and $x \in \mathbb{R}$, we let

$$U_x(f) = \inf\{t \le 0 : f(t) = x\}, \qquad T_x(f) = \inf\{t \ge 0 : f(t) = x\}.$$

In words, $U_x(f)$ is the time of the first visit to x to the left of 0, with $U_x(f) = -\infty$ if there is no such time, and $T_x(f)$ is the first time f visits x to the right of 0, with $T_x(f) = \infty$ if there is no such time. Of course, we can also apply T_x to functions in $\mathcal{C}([0,\infty),\mathbb{R})$, and U_x to functions in $\mathcal{C}((-\infty,0],\mathbb{R})$.

For $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ and $x > 0$, set

$$v(f,x) = \frac{1}{2}\big(T_{-x}(f) - U_x(f) - 2x\big)$$

whenever all terms on the right-hand side are finite, and $v(f,x) = \infty$ otherwise. Note that if x is an integer and f is the contour path of an infinite forest, then $v(f,x)$ is the total number of edges of the $2x$ trees that are encoded by f along the interval $[U_x(f), T_{-x}(f)]$.

Let $s > 0$ be given. For the rest of this section, we will always set $s_n = \lfloor (3/2) s a_n^2 \rfloor$. Since $a_n^2 \ll \sigma_n \ll n$, we will implicitly assume that n is so large such that $2s_n < \sigma_n < n$.

We first prove an absolute continuity relation on the interval $[U_{s_n}, T_{-s_n}]$ between C_n and the contour function C_∞ of a critical infinite forest. For that purpose, we define two probability laws $\mathbb{P}_{n,s}$, $\mathbb{Q}_{n,s}$ on $\mathcal{C}(\mathbb{R},\mathbb{R})$ as follows:

$$\mathbb{P}_{n,s} = \mathcal{L}\big(\big(C_n\big(\big(t \vee U_{s_n}(C_n)\big) \wedge T_{-s_n}(C_n)\big), t \in \mathbb{R}\big)\big),$$
$$\mathbb{Q}_{n,s} = \mathcal{L}\big(\big(C_\infty\big(\big(t \vee U_{s_n}(C_\infty)\big) \wedge T_{-s_n}(C_\infty)\big), t \in \mathbb{R}\big)\big).$$

LEMMA 6.17. *Let $s > 0$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, with $s_n = \lfloor (3/2) s a_n^2 \rfloor$,*

$$\sum_{f \in \operatorname{supp}(\mathbb{P}_{n,s})} \big|\mathbb{P}_{n,s}(f) - \mathrm{e}^{2s\theta - \frac{v(f,s_n)}{(9/4)a_n^4}\theta^2} \mathbb{Q}_{n,s}(f)\big| \le \varepsilon,$$

where $\operatorname{supp}(\mathbb{P}_{n,s}) \subset \mathcal{C}(\mathbb{R},\mathbb{R})$ *denotes the support of* $\mathbb{P}_{n,s}$. *Moreover, the support of* $\mathbb{P}_{n,s}$ *is contained in the support of* $\mathbb{Q}_{n,s}$, *and we have*

$$\mathbb{Q}_{n,s}\big(\operatorname{supp}(\mathbb{Q}_{n,s}) \setminus \operatorname{supp}(\mathbb{P}_{n,s})\big) = o(1) \quad \text{as } n \to \infty.$$

PROOF. From the constructions of C_n and C_∞, it is clear that each realization of $\mathbb{P}_{n,s}$ is a realization of $\mathbb{Q}_{n,s}$. Now fix $s > 0$, and let $\varepsilon > 0$. We first show that for $c_v > 0$ sufficiently large,

$$\begin{aligned} &\mathrm{e}^{2s\theta}\mathbb{Q}_{n,s}\big(f \in \mathcal{C}(\mathbb{R},\mathbb{R}) : v(f,s_n) > c_v a_n^4\big) \le \varepsilon/4 \quad \text{and} \\ &\mathbb{P}_{n,s}\big(f \in \mathcal{C}(\mathbb{R},\mathbb{R}) : v(f,s_n) > c_v a_n^4\big) \le \varepsilon/4. \end{aligned} \tag{6.30}$$

Write T_k for the first hitting time of k of a simple random walk started at zero. By construction of C_∞, we have

$$\mathbb{Q}_{n,s}\big(\{v(f,s_n) > c_v a_n^4\}\big) = \mathbb{P}\big(T_{-2s_n} > 2c_v a_n^4 + 2s_n\big),$$

and standard random walk estimates give the existence of $n_0 \in \mathbb{N}$ and $c_v > 0$ (depending on s, but s is fixed) such that for $n \ge n_0$, $\mathrm{e}^{2s\theta}\mathbb{Q}_{n,s}(\{v(f,s_n) > c_v a_n^4\}) \le \varepsilon/4$. Similarly,

$$\mathbb{P}_{n,s}\big(\{v(f,s_n) > c_v a_n^4\}\big) = \mathbb{P}\big(T_{-2s_n} > 2c_v a_n^4 + 2s_n \mid T_{-\sigma_n} = 2n + \sigma_n\big),$$

and since $\sigma_n \gg \sqrt{n}$, it is easy to check that the probability on the right is bounded by the unconditioned probability $\mathbb{P}(T_{-2s_n} > 2c_v a_n^4 + 2s_n) \le \varepsilon/4$. This shows (6.30).

Note that $f \in \operatorname{supp}(\mathbb{Q}_{n,s})\backslash\operatorname{supp}(\mathbb{P}_{n,s})$ requires $v(f,s_n) > n$. Since $a_n^4 \ll n$, it is therefore a consequence of the second part of (6.30) that

$$\mathbb{Q}_{n,s}\big(\operatorname{supp}(\mathbb{Q}_{n,s})\backslash\operatorname{supp}(\mathbb{P}_{n,s})\big) = o(1).$$

Moreover, recalling that θ and s are fixed, display (6.30) implies that for c_v sufficiently large, we have for all n large enough

$$\sum_{\substack{f \in \operatorname{supp}(\mathbb{P}_{n,s}):\\ v(f,s_n) > c_v a_n^4}} \big|\mathbb{P}_{n,s}(f) - \mathrm{e}^{2s\theta - \frac{v(f,s_n)}{(9/4)a_n^4}\theta^2}\mathbb{Q}_{n,s}(f)\big| \le \varepsilon/2.$$

It remains to argue that for fixed c_v and all n large enough, we have also

$$\sum_{\substack{f \in \operatorname{supp}(\mathbb{P}_{n,s}):\\ v(f,s_n) \le c_v a_n^4}} \big|\mathbb{P}_{n,s}(f) - \mathrm{e}^{2s\theta - \frac{v(f,s_n)}{(9/4)a_n^4}\theta^2}\mathbb{Q}_{n,s}(f)\big| \le \varepsilon/2. \tag{6.31}$$

In this regard, consider a sequence $f_n \in \mathcal{C}(\mathbb{R},\mathbb{R})$ of functions in the support of $\mathbb{P}_{n,s}$ such that $v_n = v(f_n,s_n) \le c_v a_n^4$. Let

$$x_n = \sigma_n - 2s_n, \qquad y_n = 2(n - v_n) + \sigma_n - 2s_n.$$

We can assume that both x_n and y_n are positive numbers. Let $(S(i), i \in \mathbb{N}_0)$ denote a simple random walk started at $S(0) = 0$. The probability $\mathbb{P}_{n,s}(f_n)$ is given by the probability to observe $2s_n$ particular trees of total size v_n as the first $2s_n$ trees in a forest of size n with σ_n trees. By (4.10) and Kemperman's formula (4.11), we obtain

$$\mathbb{P}_{n,s}(f_n) = \frac{\frac{x_n}{y_n}2^{y_n}\mathbb{P}(S(y_n)=x_n)}{\frac{\sigma_n}{2n+\sigma_n}2^{2n+\sigma_n}\mathbb{P}(S(2n+\sigma_n)=\sigma_n)}$$

$$= \frac{x_n}{y_n}\frac{2n+\sigma_n}{\sigma_n}2^{-2(v_n+s_n)}\frac{\mathbb{P}(S(y_n)=x_n)}{\mathbb{P}(S(2n+\sigma_n)=\sigma_n)}. \tag{6.32}$$

By definition of C_∞, $\mathbb{Q}_{n,s}(f_n)$ is the probability of a particular realization of $2s_n$ independent critical geometric Galton–Watson trees with v_n edges in total. Therefore, by (4.10),

$$\mathbb{Q}_{n,s}(f_n) = 2^{-2(v_n+s_n)}. \tag{6.33}$$

Moreover, by assumption on σ_n and a_n, we have uniformly in all possible choices of f_n that satisfy $v_n \le c_v a_n^4$,

$$\left|\frac{x_n}{y_n}\frac{2n+\sigma_n}{\sigma_n} - 1\right| = o(1). \tag{6.34}$$

Since $\sigma_n \gg \sqrt{n}$, the fraction of random walk probabilities in (6.32) is not controlled well enough by a standard local central limit theorem as formulated in (4.12). Instead, we use (4.13) and obtain

$$\frac{\mathbb{P}(S(y_n)=x_n)}{\mathbb{P}(S(2n+\sigma_n)=\sigma_n)}$$

$$= \exp\left(-\sum_{\ell=1}^{\infty}\frac{1}{2\ell(2\ell-1)}\left(\frac{x_n^{2\ell}}{{y_n}^{2\ell-1}} - \frac{\sigma_n^{2\ell}}{(2n+\sigma_n)^{2\ell-1}}\right)\right)(1+o(1)). \tag{6.35}$$

We now analyze the terms in the sum inside the exponential in the last display, similar to the proof of Lemma 5.3. First,

$$\begin{aligned}\frac{x_n^{2\ell}}{{y_n}^{2\ell-1}} &- \frac{\sigma_n^{2\ell}}{(2n+\sigma_n)^{2\ell-1}}\\ &= \frac{\sigma_n^{2\ell}}{(2n+\sigma_n)^{2\ell-1}}\biggl[-2\ell\frac{2s_n}{\sigma_n} + (2\ell-1)\frac{2(v_n+s_n)}{2n+\sigma_n}\\ &\qquad + O\left(\left(\frac{s_n}{\sigma_n}\right)^2\right) + O\left(\left(\frac{v_n+s_n}{2n+\sigma_n}\right)^2\right)\biggr].\end{aligned} \tag{6.36}$$

Using that $v_n + s_n = v_n(1+o(1))$, we now observe that

$$-\frac{2\ell\sigma_n^{2\ell}}{(2n+\sigma_n)^{2\ell-1}}\frac{2s_n}{\sigma_n} = \bigl(-4\ell s\theta + o(1)\bigr)\frac{\sigma_n^{2(\ell-1)}}{(2n+\sigma_n)^{2(\ell-1)}} \quad \text{and}$$

$$\frac{(2\ell-1)\sigma_n^{2\ell}}{(2n+\sigma_n)^{2\ell-1}}\frac{2(v_n+s_n)}{2n+\sigma_n} = (2\ell-1)\frac{2v_n}{(9/4)a_n^4}\bigl(\theta^2+o(1)\bigr)\frac{\sigma_n^{2(\ell-1)}}{(2n+\sigma_n)^{2(\ell-1)}}.$$

Since $\sigma_n \ll n$, we deduce from the last display that if $\ell \geq 2$, all the terms in (6.36) converge to 0 as $n \to \infty$, whereas for $\ell = 1$, the right-hand side of (6.36) is equal to $-4s\theta + \frac{2v_n}{(9/4)a_n^4}\theta^2 + o(1)$. For n large enough, $\sigma_n/(2n+\sigma_n) < 1/2$, so that each term in the sum in (6.35) is bounded by $C(1/2)^{2(\ell-1)}$ for some universal constant $C > 0$, which is summable. Therefore, by dominated convergence

$$\frac{\mathbb{P}(S(y_n) = x_n)}{\mathbb{P}(S(2n+\sigma_n) = \sigma_n)} = \exp\left(2s\theta - \frac{v_n}{(9/4)a_n^4}\theta^2\right) + o(1).$$

Note that all the error terms above do depend on f_n only through the constant c_v. Combining the last display with (6.33) and (6.34), display (6.31), and hence the claim of the lemma follow. $\square$

REMARK 6.18. Note that C_∞ is a discrete analog of the contour function of the Brownian half-plane BHP: The process $(C_\infty(i), i \in \mathbb{N}_0)$ is a simple random walk, and if $S = (S(i), i \in \mathbb{N}_0)$ denotes another (independent) simple random walk, then it is straightforward to check that

$$(C_\infty(-i), i \in \mathbb{N}) =_d \left(S(i+1) - 2\min_{0 \leq \ell \leq i+1} S(\ell) + 1, i \in \mathbb{N}\right),$$

that is, $(C_\infty(-i), i \in \mathbb{N})$ is a discrete Pitman-type transform of a simple random walk. In particular, $-U_k(C_\infty) =_d T_{-k}(S)$.

For proving Theorem 3.4, it is convenient to introduce some more notation. Let us first define rescaled versions $C_{n,s}$ and $\mathfrak{L}_{n,s}$ of the contour and label functions C_n and $\mathfrak{L}_n$ that capture the information encoded by the first $s_n = \lfloor (3/2)sa_n^2 \rfloor$ trees $(\tau_0, \dots, \tau_{s_n-1})$ and the last s_n trees $(\tau_{\sigma_n - s_n}, \dots, \tau_{\sigma_n - 1})$ of $\mathfrak{f}_n$,

$$(C_{n,s}(t), t \in \mathbb{R}) = \left(\frac{1}{(3/2)a_n^2} C_n\big(((9/4)a_n^4 t \vee U_{s_n}(C_n)) \wedge T_{-s_n}(C_n)\big), t \in \mathbb{R}\right),$$

$$(\mathfrak{L}_{n,s}(t), t \in \mathbb{R}) = \left(\frac{1}{a_n} \mathfrak{L}_n\big(((9/4)a_n^4 t \vee U_{s_n}(C_n)) \wedge T_{-s_n}(C_n)\big), t \in \mathbb{R}\right).$$

Let $((\mathfrak{f}_\infty, \mathfrak{l}_\infty), \mathsf{b}_\infty)$ encode the UIHPQ, with C_∞ and $\mathfrak{L}_\infty$ denoting the associated contour and label functions. In analogy to the last display, we define two random functions $C_{n,s}^\infty$ and $\mathfrak{L}_{n,s}^\infty$ from $\mathbb{R}$ to $\mathbb{R}$ by setting

$$(C_{n,s}^\infty(t), t \in \mathbb{R}) = \left(\frac{1}{(3/2)a_n^2} C_\infty\big(((9/4)a_n^4 t \vee U_{s_n}(C_\infty)) \wedge T_{-s_n}(C_\infty)\big), t \in \mathbb{R}\right),$$

$$(\mathfrak{L}_{n,s}^\infty(t), t \in \mathbb{R}) = \left(\frac{1}{a_n} \mathfrak{L}_\infty\big(((9/4)a_n^4 t \vee U_{s_n}(C_\infty)) \wedge T_{-s_n}(C_\infty)\big), t \in \mathbb{R}\right).$$

Recall the definition of the contour and label functions $X^\theta = (X^\theta(t), t \in \mathbb{R})$ and $W^\theta = (W^\theta(t), t \in \mathbb{R})$ which encode the Brownian half-plane BHP_θ (in the notation used in Section 6.2.1). We set

$$X^{\theta,s} = (X^{\theta,s}(t), t \in \mathbb{R}) = (X^{\theta}((t \vee U_s(X^{\theta})) \wedge T_{-s}(X^{\theta})), t \in \mathbb{R}),$$
$$W^{\theta,s} = (W^{\theta,s}(t), t \in \mathbb{R}) = (W^{\theta}((t \vee U_s(X^{\theta})) \wedge T_{-s}(X^{\theta})), t \in \mathbb{R}).$$

Finally, for $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, put

$$\lambda_{n,s}(f) = \exp\left(2s\theta - \frac{v(f, s_n)}{(9/4)a_n^4}\theta^2\right).$$

PROOF OF THEOREM 3.4. Let $r \geq 0$. By Lemma 5.6, our claim follows if we show that

$$B_r^{(0)}(a_n^{-1} \cdot Q_n^{\sigma_n}) \xrightarrow[n\to\infty]{(d)} B_r(\mathsf{BHP}_\theta)$$

in distribution in $\mathbb{K}$, where we recall that $\theta = \lim_{n\to\infty}(3/2)a_n^2\sigma_n/2n$. For $n \in \mathbb{N}$ and $s > 0$, define the events

$$\mathcal{E}^1(n, s) = \left\{\min_{[0,s_n]} \mathsf{b}_n < -3ra_n, \min_{[\sigma_n - s_n, \sigma_n - 1]} \mathsf{b}_n < -3ra_n\right\}$$
$$\cap \left\{\min_{[s_n+1, \sigma_n-(s_n+1)]} \mathsf{b}_n < -3ra_n\right\}$$

and similarly

$$\mathcal{E}^2(n, s) = \left\{\min_{[0,s_n]} \mathsf{b}_\infty < -3ra_n, \min_{[-s_n,0]} \mathsf{b}_\infty < -3ra_n\right\},$$
$$\mathcal{E}^3(s) = \left\{\min_{[0,s]} \gamma < -3r, \min_{[-s,0]} \gamma < -3r\right\}.$$

Let $\varepsilon > 0$ be given. Applying Lemma 5.4, we find $n_0 \in \mathbb{N}$ and $s > 0$ sufficiently large such that for $n \geq n_0$, $\mathbb{P}(\mathcal{E}^1(n, s)) \geq 1 - \varepsilon$. For possibly larger values of n and s, Donsker's invariance principle shows that also $\mathbb{P}(\mathcal{E}^2(n, s)) \geq 1 - \varepsilon$, and standard properties of Brownian motion give $\mathbb{P}(\mathcal{E}^3(s)) \geq 1 - \varepsilon$ for s large enough. We now fix $s > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, each of the events $\mathcal{E}^1, \mathcal{E}^2, \mathcal{E}^3$ has probability at least $1 - \varepsilon$.

As in the proof of Proposition 3.11, we write $\tau(\mathfrak{f}_\infty, k)$ for the tree of $\mathfrak{f}_\infty$ which is attached to (k), $k \in \mathbb{Z}$. We identify $V(\mathfrak{f}_\infty)$ with $V(Q_\infty^\infty)$, as usual. Recall that the root ρ of UIHPQ is at distance at most $-\mathsf{b}_\infty(-1) + 1$ away from (0). On the event $\mathcal{E}^2(n, s)$, the cactus bound (4.6) thus gives for vertices $v \in V(Q_\infty^\infty)$ which do not belong to any of the trees $\tau(\mathfrak{f}_\infty, k)$, $k = -s_n, \dots, s_n$,

$$d_\infty(0, v) \geq -\max\left\{\min_{[0,s_n]} \mathsf{b}_\infty, \min_{[-s_n,0]} \mathsf{b}_\infty\right\} \geq 3ra_n$$

for large n. Since for vertices u, v in $B_{ra_n}^{(0)}(Q_\infty^\infty)$, any geodesic between u and v in Q_∞^∞ lies entirely in $B_{2ra_n}^{(0)}(Q_\infty^\infty)$, we obtain from the construction of edges in the Bouttier–Di Francesco–Guitter mapping that the submap $B^{(0)}_{ra_n}(Q^{\infty}_{\infty})$ is a measurable function of $(C^{\infty}_{n,s}, \mathfrak{L}^{\infty}_{n,s})$. A similar argument which we leave to the reader (see also the first part of the proof of Proposition 3.11) shows that on $\mathcal{E}^1(n,s)$, the submap $B^{(0)}_{ra_n}(Q^{\sigma_n}_n)$ is given by the *same* function of $(C_{n,s}, \mathfrak{L}_{n,s})$. Moreover, on $\mathcal{E}^3(s)$, $B_r(\mathsf{BHP})$ is determined by $(X_{0,s}, W_{0,s})$.

By Lemma 5.5, recalling that $a_n^2 \ll \sigma_n$, we have for large n

$$\begin{aligned}&\big\|\mathcal{L}\big(\big(\mathsf{b}_n(\sigma_n - s_n), \dots, \mathsf{b}_n(\sigma_n - 1), \mathsf{b}_n(0), \mathsf{b}_n(1), \dots, \mathsf{b}_n(s_n)\big)\big)\\ &\quad - \mathcal{L}\big(\big(\mathsf{b}_\infty(-s_n), \dots, \mathsf{b}_\infty(-1), \mathsf{b}_\infty(0), \mathsf{b}_\infty(1), \dots, \mathsf{b}_\infty(s_n)\big)\big)\big\|_{\mathrm{TV}} \le \varepsilon.\end{aligned}$$

Combining this bound with Lemma 6.17, the above observations entail that for any measurable and bounded $F : \mathcal{C}(\mathbb{R},\mathbb{R})^2 \times \mathbb{K} \to \mathbb{R}$ and n large enough

$$\begin{aligned}&\big|\mathbb{E}\big[F\big(C_{n,s}, \mathfrak{L}_{n,s}, B^{(0)}_r\big(a_n^{-1} \cdot Q^{\sigma_n}_n\big)\big)\mathbb{1}_{\mathcal{E}^1(n,s)}\big]\\ &\quad - \mathbb{E}\big[\lambda_{n,s}(C_\infty) F\big(C^{\infty}_{n,s}, \mathfrak{L}^{\infty}_{n,s}, B^{(0)}_r\big(a_n^{-1} \cdot Q^{\infty}_{\infty}\big)\big)\mathbb{1}_{\mathcal{E}^2(n,s)}\big]\big| \le C\varepsilon,\end{aligned} \tag{6.37}$$

where $C > 0$ is a constant that depends only on F and θ, s, which are fixed. Recall from the proof of Lemma 6.17 that for each $\delta > 0$, we find $c_\delta > 0$ such that $\mathbb{P}(v(C_\infty, s_n) > c_\delta a_n^4) \le \delta$. Keeping in mind Remark 6.18, the joint convergence (6.29) thus implies

$$\big(C^{\infty}_{n,s}, \mathfrak{L}^{\infty}_{n,s}, B^{(0)}_r\big(a_n^{-1} \cdot Q^{\infty}_{\infty}\big)\big) \xrightarrow[n\to\infty]{(d)} \big(X^{0,s}, W^{0,s}, B_r(\mathsf{BHP})\big)$$

in $\mathcal{C}(\mathbb{R},\mathbb{R})^2 \times \mathbb{K}$, and

$$\frac{v(C_\infty, s_n)}{(9/4)a_n^4} \xrightarrow[n\to\infty]{(d)} \frac{1}{2}(T_{-s} - U_s)\big(X^0\big),$$

where, in hopefully obvious notation, X^0 stands for the contour function of the Brownian half-plane BHP with zero skewness, and $X^{0,s}$,$W^{0,s}$ were defined above in terms of BHP. For large n, we can therefore ensure that

$$\begin{aligned}&\big|\mathbb{E}\big[\lambda_{n,s}(C_\infty) F\big(C^{\infty}_{n,s}, \mathfrak{L}^{\infty}_{n,s}, B^{(0)}_r\big(a_n^{-1} \cdot Q^{\infty}_{\infty}\big)\big)\big]\\ &\quad - \mathbb{E}\big[\exp\big(2s\theta - (T_{-s} - U_s)\big(X^0\big)\theta^2/2\big) F\big(X^{0,s}, W^{0,s}, B_r(\mathsf{BHP})\big)\big]\big| \le \varepsilon.\end{aligned} \tag{6.38}$$

We will now rewrite the second expectation in the last display using Girsanov's (and implicitly Pitman's) transform. More specifically, an application of Girsanov's theorem for Brownian motion with drift $-\theta$ (see, e.g., [26], Chapter 3.5 Part C) to the right part $(X^{\theta,s}(t), t \ge 0)$ as well as to the left part $(X^{\theta,s}(t), t \le 0)$ (recall that the time-reversal of this left part is the Pitman transform of such a Brownian motion with drift, so that $-U_s$ is the first hitting time of $-s$ for this Brownian motion) shows that for $G : \mathcal{C}(\mathbb{R},\mathbb{R}) \to \mathbb{R}$ continuous and bounded,

$$\mathbb{E}\big[\exp\big(2s\theta - (T_{-s} - U_s)\big(X^0\big)\theta^2/2\big) G\big(X^{0,s}\big)\big] = \mathbb{E}\big[G\big(X^{\theta,s}\big)\big].$$

Since on the event $\mathcal{E}^3(s)$, $B_r(\mathsf{BHP})$ is a measurable function of $(X^{0,s}, W^{0,s})$ (and $B_r(\mathsf{BHP}_\theta)$ is given by the *same* measurable function of $(X^{\theta,s}, W^{\theta,s})$), we obtain

$$\begin{aligned}
&\mathbb{E}\big[\exp\big(2s\theta - (T_{-s} - U_s)(X^0)\theta^2/2\big) F\big(X^{0,s}, W^{0,s}, B_r(\mathsf{BHP})\big)\mathbb{1}_{\mathcal{E}^3(s)}\big] \\
(6.39)\qquad &= \mathbb{E}\big[F\big(X^{\theta,s}, W^{\theta,s}, B_r(\mathsf{BHP}_\theta)\big)\mathbb{1}_{\mathcal{E}^3(s)}\big].
\end{aligned}$$

Using that the three events $\mathcal{E}^1(n,s)$, $\mathcal{E}^2(n,s)$ and $\mathcal{E}^3(s)$ have all probability at least $1-\varepsilon$, a combination of (6.37), (6.38) and (6.39) shows that for large n

$$\big|\mathbb{E}\big[F\big(C_{n,s}, \mathfrak{L}_{n,s}, B_r^{(0)}\big(a_n^{-1}\cdot Q_n^{\sigma_n}\big)\big)\big] - \mathbb{E}\big[F\big(X^{\theta,s}, W^{\theta,s}, B_r(\mathsf{BHP}_\theta)\big)\big]\big| \le C'\varepsilon$$

for some C' depending only on F and s,θ. Clearly, this implies our claim. $\square$

6.6. *Coupling of Brownian disks.* We aim at showing Theorem 3.12 and Corollary 3.13. The main ideas are similar to those of Section 6.2, but closer in spirit to [20]. We begin with showing how Theorem 3.12 implies that IBD_σ is homeomorphic to the pointed closed disk $\overline{\mathbb{D}}\setminus\{0\}$.

PROOF OF COROLLARY 3.13. The arguments are similar to the proof of Corollary 3.8. First, Theorem 3.12 shows that with probability 1, for every $r>0$, the ball $B_r(\mathsf{IBD}_\sigma)$ is contained in an open set of IBD_σ homeomorphic to $\overline{\mathbb{D}}\setminus\{0\}$. In particular, IBD_σ is a noncompact surface with a boundary homeomorphic to the circle $\mathbb{S}^1$, and it has only one end. Let us glue a copy D of $\overline{\mathbb{D}}$ along the boundary of IBD_σ, hence obtaining a noncompact surface S without boundary, which is now simply connected. This surface is thus homeomorphic to $\mathbb{R}^2$. Again, the Jordan–Schoenflies theorem shows that any homeomorphism from the boundary of IBD_σ to $\mathbb{S}^1$ can be extended to a homeomorphism from S to $\mathbb{R}^2$, and this homeomorphism must send IBD_σ to the unbounded region $\{z:|z|\ge 1\}$, which in turn is homeomorphic to $\overline{\mathbb{D}}\setminus\{0\}$, as wanted. $\square$

For proving Theorem 3.12, we first collect some notation. Throughout this section, $\sigma\in(0,\infty)$ is fixed, and T denotes always a strictly positive real.

6.6.1. *Notation*: (*infinite-volume*) *Brownian disk.* We again assume that the following processes are defined on a joint probability space:

- F a first passage Brownian bridge on $[0,T]$ from 0 to $-\sigma$;
- b a Brownian bridge on $[0,\sigma]$ from 0 to 0, multiplied by $\sqrt{3}$, independent of F;
- B a standard Brownian motion started from $B_0=0$, independent of b;
- R, R' two independent three-dimensional Bessel processes with $R_0=R'_0=0$, independent of b;
- U_0 a uniform random variable in $[0,\sigma]$, independent of (F,b,B,R,R').

We define the Brownian disk $\mathsf{BD}_{T,\sigma}$ in terms of the processes F and b and use the notation for Section 6.2.1. In particular, $Z = Z^{F-\underline{F}}$ denotes the random snake driven by $F - \underline{F}$, and the label process is given by $W_t = b_{-\underline{F}_t} + Z_t$, $0 \le t \le T$.

As for the infinite-volume Brownian disk IBD_σ, we will work with the representation of the contour process given in Remark 2.9 (and denoted Y^σ there). We define it in terms of the Bessel processes R and R' and the Brownian motion B stopped at times T_{U_0} and T_σ. We will moreover write $Z^{\mathrm{I}} = Z^{Y^\sigma - \underline{\underline{Y}}^\sigma}$ for the random snake driven by $Y^\sigma - \underline{\underline{Y}}^\sigma$ (see Definition 2.8), and $W_t^{\mathrm{I}} = b_{-\underline{\underline{Y}}_t^\sigma} + Z^{\mathrm{I}}$, $t \in \mathbb{R}$, for the label process associated with IBD_σ. Note that we use the same bridge b in the definition of $\mathsf{BD}_{T,\sigma}$ and IBD_σ.

We now establish a coupling between the processes encoding $\mathsf{BD}_{T,\sigma}$ and IBD_σ, similar to Section 6.2 above.

6.6.2. *Coupling of contour functions*. It will be convenient to write $T_x = \inf\{t \ge 0 : B_t < -x\}$ for the first hitting time of $(-\infty, -x)$ of the Brownian motion B, so that $(T_x, 0 \le x \le \sigma)$ is a stable subordinator of index $1/2$ and Laplace exponent $-\log \mathbb{E}[\exp(-\lambda T_1)] = \sqrt{2\lambda}$. Recall that the density of T_x is denoted by $g_\cdot(x)$.

We may write the jump sizes of $(T_x, 0 \le x \le \sigma)$, together with the times in $[0, \sigma]$ at which they occur, as a point measure

$$\mathcal{M} = \sum_{i \ge 1} \delta_{(\Delta_i, U_i)},$$

so that $T_{U_i} - T_{U_i-} = \Delta_i$. By well-known properties of subordinators, this measure is Poisson with intensity measure $(2\pi y^3)^{-1/2}\,\mathrm{d}y \otimes \mathrm{d}u\mathbb{1}_{[0,\sigma]}(u)$. The first passage bridge consists in the process $(B_t, 0 \le t \le T)$ conditioned on the event $\{T_\sigma = T\} = \{\sum_i \Delta_i = T\}$. In order to describe the conditional law of $\mathcal{M}$, we follow Pitman ([36], Chapter 4) and fix the ordering $\Delta_1, \Delta_2, \dots$ as the *size-biased ordering* of the jumps, so that conditionally given $(\Delta_1, \dots, \Delta_i)$, Δ_{i+1} is chosen from all the remaining jumps with probability that is proportional to its size. The random variables U_i, $i \ge 1$, are then i.i.d. uniform in $[0, \sigma]$ and independent of $(\Delta_1, \Delta_2, \dots)$. This property will remain true when we condition the measure $\mathcal{M}$ on events that involve only the sequence $(\Delta_1, \Delta_2, \dots)$.

The following lemma is a consequence of [36], Lemma 4.1.

LEMMA 6.19. *Conditionally given $\{T_\sigma = T\}$, the law of Δ_1 is*

$$\begin{aligned}\mathbb{P}(\Delta_1 \in \mathrm{d}y \mid T_\sigma = T) &= \frac{\sigma\,\mathrm{d}y}{T(2\pi y)^{1/2}} \frac{g_{T-y}(\sigma)}{g_T(\sigma)} \\ &= \mathrm{e}^{\sigma^2/2T} \sqrt{\frac{T}{y}}\, g_{T-y}(\sigma)\,\mathrm{d}y,\end{aligned}$$

and given $\{T_\sigma = T, \Delta_1 = y\}$, the remaining jumps $(\Delta_2, \Delta_3, \dots)$ have the same distribution as $(\Delta_1, \Delta_2, \dots)$ conditionally given $\{T_\sigma = T - y\}$.

This allows us to obtain the main technical lemma of this section, which one should see as the continuum version of Lemmas 5.1 and 5.2: it says that given, $T_\sigma = T$, the jumps behave as those of the unconditioned subordinator $(T_x, 0 \leq x \leq \sigma)$, with the exception of the largest jump of size approximately T.

LEMMA 6.20. (a) *For every* $\delta \in (0,1)$, *one has*

$$\liminf_{T\to\infty} \mathbb{P}\Big(\Delta_1 > (1-\delta)T \,\Big|\, \sum_i \Delta_i = T\Big) = 1.$$

(b) *One has*

$$\lim_{T\to\infty} \Big\| \mathcal{L}\Big(\Delta_2, \Delta_3, \dots \,\Big|\, \sum_i \Delta_i = T\Big) - \mathcal{L}(\Delta_1, \Delta_2, \dots)\Big\|_{\mathrm{TV}} = 0.$$

PROOF. From the description of the conditional law of Δ_1 provided by Lemma 6.19, we obtain

$$\mathbb{P}\Big(\Delta_1 > (1-\delta)T \,\Big|\, \sum_i \Delta_i = T\Big) = \mathrm{e}^{\sigma^2/2T} \int_0^{\delta T} \mathrm{d}x \sqrt{\frac{T}{T-x}} g_x(\sigma)\,\mathrm{d}x,$$

and, as $T \to \infty$, the latter expression converges to $\int_0^\infty g_x(\sigma)\,\mathrm{d}x = 1$ by dominated convergence, since $\sqrt{T/(T-x)} \leq (1-\delta)^{-1/2}$. This proves (a). For (b), one can use the second part of Lemma 6.19 to obtain the disintegration

$$\begin{aligned}
&\mathcal{L}\Big(\Delta_2, \Delta_3, \dots \,\Big|\, \sum_i \Delta_i = T\Big) \\
&\qquad = \int_0^T \mathrm{d}x \mathrm{e}^{\sigma^2/2T} \sqrt{\frac{T}{T-x}} g_x(\sigma) \mathcal{L}\Big(\Delta_1, \Delta_2, \dots \,\Big|\, \sum_i \Delta_i = x\Big).
\end{aligned}$$

Since $g_\cdot(\sigma)$ is the density function of $T_\sigma = \sum_i \Delta_i$, we also have the disintegration

$$\mathcal{L}(\Delta_1, \Delta_2, \dots) = \int_0^\infty \mathrm{d}x g_x(\sigma) \mathcal{L}\Big(\Delta_1, \Delta_2, \dots \,\Big|\, \sum_i \Delta_i = x\Big),$$

which entails that

$$\begin{aligned}
&\Big\| \mathcal{L}\Big(\Delta_2, \Delta_3, \dots \,\Big|\, \sum_i \Delta_i = T\Big) - \mathcal{L}(\Delta_1, \Delta_2, \dots)\Big\|_{\mathrm{TV}} \\
&\qquad \leq \int_T^\infty g_x(\sigma)\,\mathrm{d}x + \int_0^T \Big| \mathrm{e}^{\sigma^2/2T} \sqrt{\frac{T}{T-x}} - 1 \Big| g_x(\sigma)\,\mathrm{d}x.
\end{aligned}$$

The first integral obviously converges to 0, and we can split the second integral at $T/2$ and rewrite it, after simple manipulations, as

$$\int_0^{T/2} \Big| \mathrm{e}^{\sigma^2/2T} \sqrt{\frac{T}{T-x}} - 1 \Big| g_x(\sigma)\,\mathrm{d}x + T \int_0^{1/2} \Big| \mathrm{e}^{\sigma^2/2T} \sqrt{\frac{1}{x}} - 1 \Big| g_{T(1-x)}(\sigma)\,\mathrm{d}x.$$

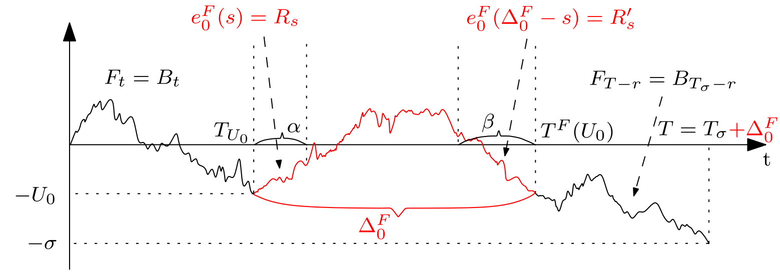

FIG. 10. *The coupling of contour functions stated as Proposition 6.21, with* $s = s(t) = t - T_{U_0}$, $r = r(t) = T - t$.

The first term converges to 0 by dominated convergence, and the second vanishes as well since $g_{T(1-x)}(\sigma) \le 2\sigma/\sqrt{\pi T^3}$ for every $x \in [0, 1/2]$. $\square$

Recall the definitions of F, B, R, R', U_0 from Section 6.6.1. We let $T^F(x) = \inf\{t \ge 0 : F_t < -x\} \wedge T$ for $0 \le x \le \sigma$. Similarly, we let $\Delta_0^F, \Delta_1^F, \Delta_2^F, \dots$ be the jump sizes of T^F ranked in size-biased order, and $U_0^F, U_1^F, \dots$ be the corresponding times. For $i \ge 0$, we let

$$e_i^F(t) = U_i^F + F\big(T^F(U_i^F-) + t\big), \qquad 0 \le t \le \Delta_i^F,$$

be the excursion of F above level $-U_i^F$; note that $\Delta_i^F = T^F(U_i^F) - T^F(U_i^F-)$.

We also let $\Delta_1, \Delta_2, \dots$ be the jump sizes of the first-hitting time subordinator $(T_x, 0 \le x \le \sigma)$. Figure 10 illustrates the following.

PROPOSITION 6.21. *For every $\varepsilon \in (0,1)$ and $\alpha, \beta > 0$, there exists $T^0 > 0$ such that for every $T \ge T^0$, it is possible to couple F, B, R, R', U_0 on the same probability space in such a way that with probability at least $1 - \varepsilon$, one has $U_0 = U_0^F$ and*

$$F_t = B_t, \qquad 0 \le t \le T^F(U_0-) = T_{U_0},$$

$$F_{T-t} = B_{T_\sigma - t}, \qquad 0 \le t \le T - T^F(U_0) = T_\sigma - T_{U_0}$$

and

$$e_0^F(t) = R_t, \qquad 0 \le t \le \alpha, \qquad e_0^F(\Delta_0^F - t) = R'_t, \qquad 0 \le t \le \beta,$$

and finally

$$\inf_{[\alpha,\infty)} R \wedge \inf_{[\beta,\infty)} R' = \min_{[\alpha, \Delta_0^F - \beta]} e_0^F.$$

PROOF. By Lemma 6.20, for T large enough, say $T > T^1$, it is possible to couple two sequences $\Delta_1, \Delta_2, \dots$ and $\Delta'_0, \Delta'_1, \Delta'_2, \dots$ on the same probability space such that:

- $(\Delta_1, \Delta_2, \dots)$ has the law of the jump sizes of $(T_x, 0 \le x \le \sigma)$ ranked in size-biased order, and
- $(\Delta'_0, \Delta'_1, \Delta'_2, \dots)$ has the law of the sequence $(\Delta_1, \Delta_2, \dots)$ conditionally given $\sum_{i\ge 1} \Delta_i = T$,

in such a way that on an event $\mathcal{E}_1$ of probability at least $1 - \varepsilon/2$, one has

$$\Delta_i = \Delta'_i, \qquad i \ge 1, \quad \text{and} \quad \Delta'_0 > T/2.$$

Extending the probability space if necessary, we can assume that it also supports an independent family of random variables $e_0, e_1, e_2, \dots$ that are independent normalized Brownian excursions, and $U_0, U_1, U_2, \dots$ that are independent uniform random variables in $[0, \sigma]$, independent of all the rest.

By Itô's synthesis of Brownian motion from its excursions, if we set, for $i \ge 1$,

$$B_t = -U_i + \sqrt{\Delta_i} e_i\biggl(\biggl(t - \sum_{j:U_j<U_i} \Delta_j\biggr)\Big/\Delta_i\biggr),$$

whenever $\sum_{j\ge 1:U_j<U_i} \Delta_i < t \le \sum_{j\ge 1:U_j\le U_i} \Delta_j$, then this a.s. extends to a continuous path $(B_t, 0 \le t \le \sum_{i\ge 1} \Delta_i)$ which is a trajectory of Brownian motion stopped when first hitting $-\sigma$, which occurs at time $T_\sigma = \sum_{i\ge 1} \Delta_i$. Similarly, setting, this time for $i \ge 0$,

$$F(t) = -U_i + \sqrt{\Delta'_i} e_i\biggl(\biggl(t - \sum_{j:U_j<U_i} \Delta'_j\biggr)\Big/\Delta'_i\biggr),$$

whenever $\sum_{j\ge 0:U_j<U_i} \Delta'_i < t \le \sum_{j\ge 0:U_j\le U_i} \Delta'_j$, this extends to a trajectory of a first passage bridge $(F(t), 0 \le t \le T)$ from 0 to $-\sigma$, as the notation suggests, and if we set $\Delta_i^F = \Delta'_i$ for $i \ge 0$, then, by definition of F, $(\Delta_i^F, i \ge 0)$ is indeed a size-biased ordering of the jumps of the first hitting time process of negative values of F.

On the event $\mathcal{E}_1$, the two processes B and F coincide on $[0, \sum_{j\ge 1:U_j<U_0} \Delta_j]$, and likewise, $B_{T_\sigma -\cdot}$ and $F(T - \cdot)$ coincide on $[0, \sum_{j\ge 1:U_j>U_0} \Delta_j]$. This yields the first displayed identity in the statement, since by construction

$$\sum_{j\ge 1:U_j<U_0} \Delta'_j = T^F(U_0-), \qquad \sum_{j\ge 1:U_j<U_0} \Delta_j = \sum_{j\ge 1:U_j\le U_0} \Delta_j = T_{U_0},$$

while we have

$$\sum_{j\ge 1:U_j>U_0} \Delta'_j = T - T^F(U_0), \qquad \sum_{j\ge 1:U_j>U_0} \Delta_j = \sum_{j\ge 1:U_j\ge U_0} \Delta_j = T_\sigma - T_{U_0}.$$

Finally, in this construction, and still in restriction to $\mathcal{E}_1$, $e_0^F = e_0(\Delta'_0\cdot)/\sqrt{\Delta'_0}$ is an excursion of Brownian motion with duration $\Delta'_0 > T/2$. At this point, we can apply Proposition 3 in [20], in the same way as in the proof of Proposition 4 therein. Up to a further extension of the probability space, as soon as T is chosen large enough, say $T > T^2$, we can couple this "long" excursion with two independent Bessel processes R, R' (and independent of all previously defined random variables) in such a way that the three last identities of the statement are satisfied on an event $\mathcal{E}_2$ with probability at least $1 - \varepsilon/2$. This yields the wanted result with $T^0 = T^1 \vee T^2$, since the intersection $\mathcal{E}_1 \cap \mathcal{E}_2$ has probability at least $1 - \varepsilon$. □

6.6.3. *Isometry of balls in* $\mathsf{BD}_{T,\sigma}$ *and* IBD_σ. As in Section 6.2, we first prove the following simplification of Theorem 3.12.

PROPOSITION 6.22. *Fix $\sigma \in (0, \infty)$, and let $\varepsilon > 0$, $r \geq 0$. There exists $T_0 = T_0(\varepsilon, r, \sigma)$ such that for all $T \geq T_0$, we can construct copies of $\mathsf{BD}_{T,\sigma}$ and IBD_σ on the same probability space such that with probability at least $1 - \varepsilon$, the balls $B_r(\mathsf{BD}_{T,\sigma})$ and $B_r(\mathsf{IBD}_\sigma)$ of radius r around the respective roots are isometric.*

With the coupling from the preceding section at hand, the proof of the proposition is a minor modification of [20], Proof of Proposition 4 (compare also with Proposition 6.6 and its proof). We will point at the necessary modifications and leave it to the reader to fill in the remaining details.

PROOF. We fix $\sigma \in (0, \infty)$, $\varepsilon > 0$ and let $r \geq 0$. We work with the notation and with the processes specified in Section 6.6.1. Let us first introduce a few events. For $K > 0$, put

$$\mathcal{E}^1(K) = \Big\{ \max_{[0,\sigma]} b < K \Big\}.$$

Then, given $A > 0$, with $\zeta = (\zeta_t, t \geq 0)$ denoting a Brownian motion started at 0, let

$$\mathcal{E}^2(\zeta, A, K) = \Big\{ \min_{[0,A]} \zeta < -10r - K, \min_{[A,A^2]} \zeta < -10r - K, \min_{[A^2,A^4]} \zeta < -10r - K \Big\},$$

and for $A > 0$ and $\alpha > 0$, set

$$\mathcal{E}^3(A, \alpha) = \Big\{ \inf_{[\alpha,\infty)} R \wedge \inf_{[\alpha,\infty)} R' > A^4 \Big\}.$$

We first choose K sufficiently large such that $\mathbb{P}(\mathcal{E}^1) \geq 1 - \varepsilon/6$. Then standard properties of Brownian motion allow us to find $A > 0$ such that $\mathbb{P}(\mathcal{E}^2) \geq 1 - \varepsilon/6$ as well, and with such a fixed A, we find by transience of the Bessel process an $\alpha > 0$ large enough such that $\mathbb{P}(\mathcal{E}^3) > 1 - \varepsilon/3$.

Recall from Section 6.6.1 and Remark 2.9 the construction of the contour process Y^σ of IBD_σ. On the coupling event $\mathcal{E}^4 = \mathcal{E}^4(\alpha, T)$ described in the statement of Proposition 6.21 (with $\beta = \alpha$), we obtain that

$$F_t = Y_t^\sigma \qquad \text{for } t \in [0, T_{U_0} + \alpha],$$
$$F_{T-t} + \sigma = Y_{-t}^\sigma \qquad \text{for } t \in [0, T_\sigma - T_{U_0} + \alpha].$$

We now work always conditionally on (F, B, R, R', U_0). Recall that $Z = Z^{F-\underline{F}}$ and $Z^{\mathrm{I}} = Z^{Y^\sigma - \underline{\underline{Y}}^\sigma}$ are the random snakes driven by $F - \underline{F}$ and $Y^\sigma - \underline{\underline{Y}}^\sigma$, respectively. Similar to the considerations around (6.15) and (6.16) in the proof of Proposition 6.6, one checks that on the event $\mathcal{E}^4$, the conditional covariance function knowing F of

$$\big(Z_t, 0 \le t \le T_{U_0} + \alpha\big), \qquad \big(Z_{T-t}, 0 \le t \le T_\sigma - T_{U_0} + \alpha\big)$$

is the same as the conditional covariance knowing Y^σ of

$$\big(Z^{\mathrm{I}}_t, 0 \le t \le T_{U_0} + \alpha\big), \qquad \big(Z^{\mathrm{I}}_{T-t}, 0 \le t \le T_\sigma - T_{U_0} + \alpha\big).$$

(Note that the special definition of the snake when the driving function is indexed by $(-\infty, \infty)$ is used in a crucial way here; see Section 2.2.) Consequently, we may assume that Z and Z^{I} are coupled such that $Z_t = Z^{\mathrm{I}}_t$ for $t \in [0, T_{U_0} + \alpha]$, and $Z_{T-t} = Z^{\mathrm{I}}_{-t}$ for $t \in [0, T_\sigma - T_{U_0} + \alpha]$. Still on the event $\mathcal{E}^4$, we have $\underline{F}_t = \underline{\underline{Y}}^\sigma_t$ for $t \in [0, T_{U_0} + \alpha]$, and $\underline{F}_{T-t} = \underline{\underline{Y}}^\sigma_{-t}$ for $t \in [0, T_\sigma - T_{U_0} + \alpha]$, so that for the label functions W and W^{I}, we have

$$W_t = W^{\mathrm{I}}_t \qquad \text{for } t \in [0, T_{U_0} + \alpha], \qquad W_{T-t} = W^{\mathrm{I}}_{-t} \qquad \text{for } t \in [0, T_\sigma - T_{U_0} + \alpha].$$

From Proposition 6.21, we derive that for the choice of α from above, the coupling event $\mathcal{E}^4(\alpha, T)$ has probability at least $1 - \varepsilon/3$ provided T is sufficiently large, and we shall work with such a T. The reminder of the proof is now close to [20], Proof of Proposition 4. For every $x \ge 0$, let

$$\eta'_{\mathrm{l}}(x) = \sup\big\{0 \le t \le \Delta^F_0/2 : e^F_0(t) = x\big\} + T_{U_0},$$
$$\eta'_{\mathrm{r}}(x) = \Delta^F_0 - \inf\big\{\Delta^F_0/2 \le t \le \Delta^F_0 : e^F_0(t) = x\big\} + T_\sigma - T_{U_0},$$

where we agree that $\eta'_{\mathrm{l}}(x) = -\infty$ (or $\eta'_{\mathrm{r}}(x) = -\infty$) if the supremum (or infimum) is taken over the empty set. Furthermore, let

$$\eta^{\mathrm{I}}_{\mathrm{l}}(x) = \sup\{t \ge 0 : R_t = x\} + T_{U_0},$$
$$\eta^{\mathrm{I}}_{\mathrm{r}}(x) = \sup\big\{t \ge 0 : R'_t = x\big\} + T_\sigma - T_{U_0}. \tag{6.40}$$

Then the process $(Z^{\mathrm{I}}_{\eta^{\mathrm{I}}_{\mathrm{l}}(x)}, x \ge 0)$ has the law of Brownian motion started at $Z^{\mathrm{I}}_{T_{U_0}} = 0$. Choosing this Brownian motion in the definition of the event $\mathcal{E}^2$ from above, so that on $\mathcal{E}^2$, we have

$$\min_{[0,A]} Z^{\mathrm{I}}_{\eta^{\mathrm{I}}_{\mathrm{l}}(\cdot)} < -6r - K,$$
$$\min_{[A,A^2]} Z^{\mathrm{I}}_{\eta^{\mathrm{I}}_{\mathrm{l}}(\cdot)} < -6r - K, \tag{6.41}$$
$$\min_{[A^2,A^4]} Z^{\mathrm{I}}_{\eta^{\mathrm{I}}_{\mathrm{l}}(\cdot)} < -6r - K,$$

we shall from now on work on the intersection of events

$$\mathcal{G} = \mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3 \cap \mathcal{E}^4, \tag{6.42}$$

which has probability at least $1 - \varepsilon$.

On $\mathcal{E}^3 \cap \mathcal{E}^4$, we note that $\min_{[\alpha, \Delta_0^F - \alpha]} e_0^F = \inf_{[\alpha,\infty)} R \wedge \inf_{[\alpha,\infty)} R' > A^4$, whence for $x \in [0, A^4]$, $\eta_{\mathrm{l}}'(x) = \eta_{\mathrm{l}}^{\mathrm{I}}(x) < T_{U_0} + \alpha$ and $\eta_{\mathrm{r}}'(x) = \eta_{\mathrm{r}}^{\mathrm{I}}(x) < T_\sigma - T_{U_0} + \alpha$. It follows that for any $x \in [0, A^4]$,

$$Z_{\eta_{\mathrm{l}}'(x)} = Z^{\mathrm{I}}_{\eta_{\mathrm{l}}^{\mathrm{I}}(x)} = Z^{\mathrm{I}}_{-\eta_{\mathrm{r}}^{\mathrm{I}}(x)} = Z_{T - \eta_{\mathrm{r}}'(x)}.$$

We are now almost in a setting where we can appeal to the reasoning in [20], Section 3.2. We should still adapt the definition of $\tilde{d}_W(s,t)$ given just before Lemma 6.7 to the setting considered here. Let $s, t \in [0, T]$. If s, t lie both in either $[0, T_{U_0} + \Delta_0^F/2]$ or in $[T_{U_0} + \Delta_0^F/2, T]$, we let

$$d_W'(s,t) = W_s + W_t - 2 \min_{[s\wedge t, s\vee t]} W.$$

Otherwise, we set

$$d_W'(s,t) = W_s + W_t - 2 \min_{[0,s\wedge t]\cup[s\vee t, T]} W.$$

Recall the definition of the pseudo-metric $D(s,t)$ associated to the Brownian disk $\mathsf{BD}_{T,\sigma}$. The following statement replaces Lemma 6.7 and is close to [20], Lemma 5(i).

LEMMA 6.23. *Assume $\mathcal{G}$ holds.*

(a) *For every $t \in [\eta_{\mathrm{l}}'(A), T - \eta_{\mathrm{r}}'(A)]$, $D(0,t) > r$.*

(b) *For every $s, t \in [0, \eta_{\mathrm{l}}'(A)] \cup [0, T - \eta_{\mathrm{r}}'(A)]$ with $\max\{D(0,s), D(0,t)\} \le r$, it holds that*

$$D(s,t) = \inf_{s_1,t_1,\dots,s_k,t_k} \sum_{i=1}^{k} d_W'(s_i, t_i),$$

where the infimum is over all possible choices of $k \in \mathbb{N}$ and reals $s_1, \dots, s_k, t_1, \dots, t_k \in [0, \eta_{\mathrm{l}}'(A^2)] \cup [T - \eta_{\mathrm{r}}'(A^2), T]$ such that $s_1 = s$, $t_k = t$, and $d_F(t_i, s_{i+1}) = 0$ for $1 \le i \le k-1$.

PROOF. One can follow the same line of reasoning as in [20], proof of Lemma 5(i), with one small modification, which is apparent from the proof of (a), so let us prove this part. If $t \in [\eta_{\mathrm{l}}'(A), T - \eta_{\mathrm{r}}'(A)]$, then by the cactus bound (6.13),

$$D(0,t) \ge W_t - 2 \max\left\{ \min_{[\![0,t]\!]_{\mathcal{T}_F}} W, \min_{[\![t,0]\!]_{\mathcal{T}_F}} W \right\}.$$

Recalling that $W_t = b_{-\underline{F}_t} + Z_t$, we first remark that on the event $\mathcal{E}^3 \cap \mathcal{E}^4$, since $\eta'_1(A) < T_{U_0} + \alpha$, we have $Z = Z^{\mathrm{I}}$ on $[0, \eta'_1(A)]$. Since $\eta'_1([0, A]) \subset [0, \eta'_1(A)]$, it follows now from (6.41) that the minimum of Z on $[0, \eta'_1(A)]$ is bounded from above by $-6r - K$. But on $\mathcal{E}^1$, $\max b < K$, so that $\min_{[\![0,t]\!]_{\mathcal{T}_F}} W \leq \min_{[\![0,\eta'_1(A)]\!]_{\mathcal{T}_F}} W \leq -6r$. A similar argument holds for the second minimum, so that in fact $D(0, t) \geq 6r$ whenever $t \in [\eta'_1(A), T - \eta'_{\mathrm{r}}(A)]$. For (b), one can follow [20], proof of Lemma 5(i), or modify the proof of (b) in Lemma 6.7. $\square$

Entirely similar, one finds the corresponding statement for the pseudo-metric D^{I} of IBD_σ that replaces Lemma 6.8: In the statement there, η'_{r} and η'_1 have to be replaced by $\eta^{\mathrm{I}}_{\mathrm{r}}$ and η^{I}_1 as defined under (6.40), and D_θ, d_{W^θ} by D^{I} and $d_{W^{\mathrm{I}}}$. Following again [20], or adapting the second part of the proof of Proposition 6.6, these two lemmas lead to the stated isometry between $B_r(\mathsf{BD}_{T,\sigma})$ and $B_r(\mathsf{IBD}_\sigma)$ on the event $\mathcal{G}$ of probability at least $1 - \varepsilon$, completing thereby the proof of Proposition 6.22. $\square$

It remains to show how Proposition 6.22 can be improved to Theorem 3.12.

6.6.4. *Proof of Theorem* 3.12. The proof is close in spirit to that of Theorem 3.7: for a fixed $r \geq 0$, we must find some $r_0 > r$ large enough so that for all T sufficiently large, the ball $B_{r_0}(\mathsf{BD}_{T,\sigma})$ contains with high probability an open set A_{BD} homeomorphic to the pointed closed disk $\overline{\mathbb{D}} \setminus \{0\}$ which, in turn, contains the ball $B_r(\mathsf{BD}_{T,\sigma})$ with high probability. Then we will apply Proposition 6.22 to couple the balls $B_{r_0}(\mathsf{BD}_{T,\sigma})$ and $B_{r_0}(\mathsf{IBD}_\sigma)$. The set A_{BD} will be defined as a region bounded by certain geodesic paths.

We use the notation specified in Section 6.6.1 and abbreviate the Brownian disk $\mathsf{BD}_{T,\sigma}$ again by $\mathsf{Y} = ([0, T]/\{D = 0\}, D, \rho)$. As in the proof of Theorem 3.7, we denote by p_{Y} the associated canonical projection. We will also use the geodesic paths Γ_s, $s \in [0, T]$, in Y respectively from $p_{\mathsf{Y}}(s)$ to $x_* = p_{\mathsf{Y}}(s_*)$ defined around Lemma 6.14, together with the properties stated there.

We will work on the coupling event $\mathcal{G}$ given by (6.42). The parameters of this event (in particular the real A and the coupling radius r_0) will be chosen later on. Furthermore, we consider another real $a_0 > 0$; below the proof of Lemma 6.24, we will first choose a_0 and then A such that $a_0 \leq A^4$, which we will assume from now now.

We let $A^0_{\mathsf{BD}} = [0, \eta'_1(a_0)] \cup [T - \eta'_{\mathrm{r}}(a_0), T]$, where $\eta'_1, \eta'_{\mathrm{r}}$ are defined on $\mathcal{G}$ in the proof of Proposition 6.22. Since $a_0 \leq A^4$, we know on $\mathcal{G}$ that $\eta'_1(a_0) < \infty$ and the same for $\eta'_{\mathrm{r}}(a_0)$. We will moreover work on the event $\{s_* \notin A^0_{\mathsf{BD}}\}$, which holds with high probability provided T is large enough.

The set A^0_{BD} will play a role analogous to that of O^0_{BD} in the proof of Theorem 3.7. Note, however, that the points $p_{\mathsf{Y}}(\eta'_1(a_0))$ and $p_{\mathsf{Y}}(T - \eta'_{\mathrm{r}}(a_0))$ are equal, and we denote this point by x_0. From Lemma 6.11, we know that $x_0 \notin \partial\mathsf{Y}$. We let

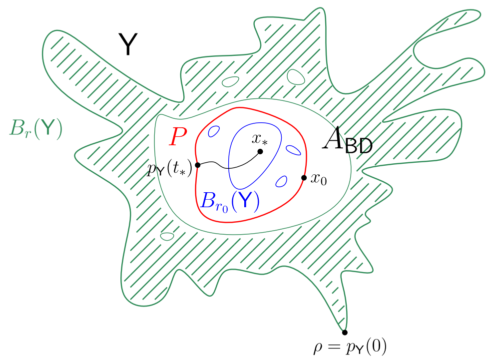

FIG. 11. *Illustration of the proof of Theorem 3.12. We look at the disk $\mathsf{Y} = \mathsf{BD}_{T,\sigma}$ from above. The ball $B_r(\mathsf{Y})$ contains the full boundary of Y and is included in the larger ball $B_{r_0}(\mathsf{Y})$, whose boundary in Y is indicated by the loops in blue. The ball $B_{r_0}(\mathsf{Y})$ encompasses the open set A_{BD}, which is homeomorphic to the pointed disk $\overline{\mathbb{D}} \setminus \{0\}$. The set A_{BD} is bordered by the boundary of Y and the simple loop P (in red), which is formed by two segments of geodesics between x_0 and $p_{\mathsf{Y}}(t_*)$.*

$t_* \in A^0_{\mathsf{BD}}$ be such that $W_{t_*} = \min_{A^0_{\mathsf{BD}}} W$. The geodesic paths $\Gamma_{\eta'_1(a_0)}$ and $\Gamma_{T-\eta'_r(a_0)}$ both start from x_0, but by Lemma 6.14(d), they become disjoint until they meet again for the first time at the point $p_{\mathsf{Y}}(t_*)$. Therefore, the segments of these geodesics between x_0 and $p_{\mathsf{Y}}(t_*)$ form a simple loop P, which is disjoint from the boundary $\partial\mathsf{Y}$ by (c) in Lemma 6.11. We point at Figure 11 for an illustration. The analog of Lemma 6.15 is the following.

LEMMA 6.24. *In the above setting, the set P is a simple loop in Y containing x_0 that does not intersect $\partial\mathsf{Y}$. Letting A_{BD} be the connected component of $\mathsf{Y} \setminus P$ that contains $p_{\mathsf{Y}}(0)$, then A_{BD} is almost surely homeomorphic to the pointed closed disk $\overline{\mathbb{D}} \setminus \{0\}$, and is the interior of the set $p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$ in Y.*

PROOF. The proof is very similar to that of Lemma 6.15. The fact that A_{BD} is a.s. homeomorphic to $\overline{\mathbb{D}} \setminus \{0\}$ is a direct consequence of the fact that Y is homeomorphic to $\overline{\mathbb{D}}$ and that P is a simple loop not intersecting $\partial\mathsf{Y}$. It only remains to be proved that A_{BD} is the interior of $p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$. However, using the paths Ξ_s defined in the proof of Lemma 6.15, it is simple to see that a point in $p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$ is linked to $\partial\mathsf{Y}$, and hence to $p_{\mathsf{Y}}(0)$, by a simple path that intersects P, if at all, only at its starting point. On the other hand, we claim that for every $x \in \mathsf{Y} \setminus p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$, we can find a simple path from x to $x_* = p_{\mathsf{Y}}(s_*)$ not intersecting P. (Note that since we are working on the event $\{s_* \notin A^0_{\mathsf{BD}}\}$, we have $x_* \notin A_{\mathsf{BD}}$.) Indeed, writing $x = p_{\mathsf{Y}}(s)$, such a path can be obtained by concatenating segments of the paths $p_{\mathsf{Y}} \circ \Xi_s$ and $p_{\mathsf{Y}} \circ \Xi_{s_*}$. We leave the details to the reader. This proves that A_{BD} and $\mathsf{Y} \setminus p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$ are the two connected components of $Y \setminus P$ and, therefore, A_{BD} is the interior of $p_{\mathsf{Y}}(A^0_{\mathsf{BD}})$. $\square$

We turn back to the proof of Theorem 3.12 and fix once for all $\sigma > 0$, $r > 0$, and $0 < \varepsilon < 1$. Recall the construction of the space IBD_σ and the definition of $\eta_{\mathrm{l}}^{\mathrm{I}}(x)$ and $\eta_{\mathrm{r}}^{\mathrm{I}}(x)$ from (6.40) in terms of its contour function. We first choose $K > 0$ so large such that the event $\mathcal{E}^1(K) = \{\max_{[0,\sigma]} b < K\}$ considered in the proof of Proposition 6.22 has probability at least $1 - \varepsilon/24$. Next, we may choose $a_0 > 0$ large enough such that

$$\mathbb{P}\Big(\min_{[0,a_0]} Z^{\mathrm{I}}_{\eta_{\mathrm{l}}^{\mathrm{I}}(\cdot)} < -2r - K\Big) \geq 1 - \varepsilon/4.$$

With $\omega(f, I) = \sup_I f - \inf_I f$, we then fix $r_0 \geq r$ large enough in such a way that

$$\mathbb{P}\big(\omega\big(W^{\mathrm{I}}, [-\eta_{\mathrm{r}}^{\mathrm{I}}(a_0), \eta_{\mathrm{l}}^{\mathrm{I}}(a_0)]\big) \leq r_0/2\big) \geq 1 - \varepsilon/4.$$

We now specify the parameters of the coupling event $\mathcal{G}$ given by (6.42): we use r_0 instead of r, the above real K, and we choose the parameter A large enough such that $A^4 \geq a_0$. We may moreover choose the remaining parameters α and T in the definition of $\mathcal{G}$ in such a way that $\mathbb{P}(\mathcal{G}) \geq 1 - \varepsilon/4$. We recall that on $\mathcal{G}$, one has $\eta_{\mathrm{l}}'(x) = \eta_{\mathrm{l}}^{\mathrm{I}}(x)$ and $\eta_{\mathrm{r}}'(x) = \eta_{\mathrm{r}}^{\mathrm{I}}(x)$ for every $x \leq A^4$, so that these equalities hold in particular whenever $x \leq a_0$ by our choice of A. By possibly taking T even larger, we can moreover ensure that the event $\{s_* \notin A^0_{\mathsf{BD}}\}$ has probability at least $1 - \varepsilon/4$. We finally work on the event

$$\mathcal{G} \cap \Big\{\min_{[0,a_0]} Z^{\mathrm{I}}_{\eta_{\mathrm{l}}^{\mathrm{I}}(\cdot)} < -2r - K\Big\} \cap \big\{\omega\big(W^{\mathrm{I}}, [-\eta_{\mathrm{r}}^{\mathrm{I}}(a_0), \eta_{\mathrm{l}}^{\mathrm{I}}(a_0)]\big) \leq r_0/2\big\} \cap \big\{s_* \notin A^0_{\mathsf{BD}}\big\},$$

which has probability at least $1 - \varepsilon$. From here on, we may follow the end of the proof of Theorem 3.7: We replace O_{BD} and $O_{\mathsf{BHP}} = I(O_{\mathsf{BD}})$ by A_{BD} and $A_{\mathsf{IBD}} = I(A_{\mathsf{BD}})$, where I is defined as before Corollary 6.9 and defines an isometry between $B_{r_0}(\mathsf{Y})$ and $B_{r_0}(\mathsf{IBD}_\sigma)$ on the coupling event $\mathcal{G}$. Then, by virtually the same arguments, we obtain that on the above intersection of events, we have $B_r(\mathsf{Y}) \subset A_{\mathsf{BD}} \subset B_{r_0}(\mathsf{Y})$. Being the image of A_{BD} under the isometric map I, A_{IBD} is itself open and homeomorphic to the pointed closed disk, and the proof of Theorem 3.12 follows.

6.7. *Infinite-volume Brownian disk.* For proving Theorem 3.2, we will combine the convergence toward the Brownian disk $\mathsf{BD}_{T,\sigma}$ proved in [11], Theorem 1 (see display (6.26)) with the couplings Theorem 3.12 and Proposition 3.14. We work in the usual setting specified in Section 4.5.4.

PROOF OF THEOREM 3.2. Let $1 \ll \sigma_n \ll \sqrt{n}$, and assume that for some $\sigma \in (0, \infty)$, $a_n \sim (4/9)^{1/4}\sqrt{\sigma_n/\sigma}$. We have to show that for each $r \geq 0$,

$$B_r\big(a_n^{-1} \cdot Q_n^{\sigma_n}\big) \xrightarrow[n\to\infty]{(d)} B_r(\mathsf{IBD}_\sigma)$$

in distribution in $\mathbb{K}$. We fix $\varepsilon > 0$ and $r \geq 0$. By Theorem 3.12, we find T_0 such that for all $T \geq T_0$, we can construct on the same probability space copies of $\mathsf{BD}_{T,\sigma}$ and IBD_σ such that with probability at least $1-\varepsilon$, we have an isometry of balls

$$B_r(\mathsf{BD}_{T,\sigma}) = B_r(\mathsf{IBD}_\sigma). \tag{6.43}$$

By Proposition 3.14, we find $R_0 \geq T_0/(2\sigma^2)$ such that for $R \geq R_0$ and n sufficiently large, we can construct on the same probability space copies of $Q_n^{\sigma_n}$ and $Q_{R\sigma_n^2}^{\sigma_n}$ such that with probability at least $1-\varepsilon$, there is the isometry

$$B_{ra_n}(Q_n^{\sigma_n}) = B_{ra_n}(Q_{R\sigma_n^2}^{\sigma_n}). \tag{6.44}$$

Now let $F : \mathbb{K} \to \mathbb{R}$ be a bounded and continuous function and $R \geq R_0$. We assume that $Q_n^{\sigma_n}$ and $Q_{R\sigma_n^2}^{\sigma_n}$ are constructed on the same probability space such that (6.44) holds, and similarly $\mathsf{BD}_{2R\sigma^2,\sigma}$ and IBD_σ so that (6.43) is satisfied. We write

$$\begin{aligned}
&\left|\mathbb{E}\left[F\left(B_r\left(a_n^{-1}\cdot Q_n^{\sigma_n}\right)\right)\right] - \mathbb{E}\left[F\left(B_r(\mathsf{IBD}_\sigma)\right)\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[F\left(a_n^{-1}\cdot B_{ra_n}(Q_n^{\sigma_n})\right) - F\left(a_n^{-1}\cdot B_{ra_n}(Q_{R\sigma_n^2}^{\sigma_n})\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[F\left(a_n^{-1}\cdot B_{ra_n}(Q_{R\sigma_n^2}^{\sigma_n})\right)\right] - \mathbb{E}\left[F\left(B_r(\mathsf{BD}_{2R\sigma^2,\sigma})\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[F\left(B_r(\mathsf{BD}_{2R\sigma^2,\sigma})\right) - F\left(B_r(\mathsf{IBD}_\sigma)\right)\right]\right|.
\end{aligned}$$

Using (6.44) and (6.43) (note that $2R\sigma^2 \geq T_0$), the first and third summand on the right-hand side are bounded from above by $2\varepsilon \sup F$. The scaling property $\lambda \cdot \mathsf{BD}_{1,\sigma} =_d \mathsf{BD}_{\lambda^4,\lambda^2\sigma}$ for $\lambda > 0$ combined with the convergence (6.26) implies that the second summand converges to zero as $n \to \infty$. This completes the proof of Theorem 3.2. $\square$

6.8. *Brownian disk limits.*

PROOF OF COROLLARY 3.15. Depending on the regime, we define the limit space $\mathcal{X}$ as in the statement of Corollary 3.15. We then have to show that for each $r \geq 0$, when T tends to infinity, $B_r(\mathsf{BD}_{T,\sigma(T)})$ converges in law to the ball of radius r around the root in $\mathcal{X}$. As usual, we consider only the case $r = 1$. Let $F : \mathbb{K} \to \mathbb{R}$ be bounded and continuous. For $T \in \mathbb{N}$ and $n \in \mathbb{N}$, we set

$$m_n(T) = Tn, \qquad \sigma_n(T) = \lfloor \sigma(T)\sqrt{2n} \rfloor, \qquad a_n = (8/9)^{1/4} n^{1/4}.$$

We write

$$\begin{aligned}
&\left|\mathbb{E}\left[F\left(B_1(\mathsf{BD}_{T,\sigma(T)})\right)\right] - \mathbb{E}\left[F\left(B_1(\mathcal{X})\right)\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[F\left(B_1(\mathsf{BD}_{T,\sigma(T)})\right)\right] - \mathbb{E}\left[F\left(a_n^{-1}\cdot B_{a_n}(Q_{m_n(T)}^{\sigma_n(T)})\right)\right]\right| \\
&\qquad + \left|\mathbb{E}\left[F\left(a_n^{-1}\cdot B_{a_n}(Q_{m_n(T)}^{\sigma_n(T)})\right)\right] - \mathbb{E}\left[F\left(B_1(\mathcal{X})\right)\right]\right|.
\end{aligned}$$

For each fixed $T \in \mathbb{N}$, [11], Theorem 1, and the scaling property of the Brownian disk imply that the first summand on the right-hand side is bounded by ε, provided $n \geq n_0(T)$ (see also (6.26) above for the case $T = 1$). We now argue by contradiction that for large enough T, there exists $n_0 = n_0 = (T, \varepsilon)$ such that for any $n \geq n_0$, the second summand is bounded by ε as well. Indeed, assuming this is not the case, we find two sequences of integers $(T_k, k \in \mathbb{N})$, $(n_k, k \in \mathbb{N})$ with $T_k \to \infty$, $n_k \to \infty$, such that

$$\left|\mathbb{E}\left[F\left(a_{n_k}^{-1} \cdot B_{a_{n_k}}\left(Q_{m_{n_k}(T_k)}^{\sigma_{n_k}(T_k)}\right)\right)\right] - \mathbb{E}\left[F\left(B_1(\mathcal{X})\right)\right]\right| > \varepsilon.$$

In the first case of the corollary where $\sigma(T) \to 0$ as $T \to \infty$ and $\mathcal{X} = \mathsf{BP}$, we have $\sqrt{\sigma_{n_k}(T_k)} \ll a_{n_k} \ll (m_{n_k}(T_k))^{1/4}$, and the last display clearly contradicts Theorem 3.1. In the second case where $\sigma(T) \to \varsigma \in (0, \infty)$, we use Theorem 3.2 instead of Theorem 3.1, and an identical argument allows us to complete the proof in this case, with $\mathcal{X}$ given by IBD_ς. In the fourth case where $\sigma(T)/T \to \infty$ and $\mathcal{X} = \mathsf{SCRT}$, we apply Theorem 3.5 instead.

Let us finally look at the third case where $\sigma(T) \to \infty$, $\sigma(T)/T \to \theta \in [0, \infty)$, and $\mathcal{X} = \mathsf{BHP}_\theta$. If $\theta = 0$, then, along sequences $(T_m, m \in \mathbb{N})$ tending to infinity for which $\sigma(T_m)/\sqrt{T_m} \to 0$ as $m \to \infty$, we follow the same argumentation by contradiction and use Theorem 3.3, whereas if $\liminf_{m\to\infty} \sigma(T_m)/\sqrt{T_m} > 0$, the corollary is a direct consequence of Theorem 3.7, and so it is in the case $\theta > 0$. □

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SUPPLEMENTARY MATERIAL

Supplement to "Classification of scaling limits of uniform quadrangulations with a boundary." (DOI: 10.1214/18-AOP1316SUPP; .pdf). We provide the proofs of Lemmas 5.1, 5.2, 5.3 and 5.5, as well as the proof of Theorem 3.5, where the SCRT appears in the limit.

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